T-79.3001 Logic in computer science: foundations **Spring 2007**
Exercise 3 (propositional logic 3.6 – 3.8)
February 6–8, 2007

Solutions to demonstration problems

4. Define Sheffer stroke using Peirce arrow.

Solution.
Definition of Sheffer stroke: $A \mid B \equiv \neg(A \wedge B)$.
Definition of Peirce arrow: $A \downarrow B \equiv \neg(A \vee B)$.

$$\begin{aligned} \neg\alpha &\equiv \alpha \downarrow \alpha. \\ (\alpha \wedge \beta) &\equiv \neg(\neg\alpha \vee \neg\beta) \equiv (\neg\alpha \downarrow \neg\beta) \equiv (\alpha \downarrow \alpha) \downarrow (\beta \downarrow \beta). \\ A \mid B \equiv \neg(\alpha \wedge \beta) &\equiv ((\alpha \downarrow \alpha) \downarrow (\beta \downarrow \beta)) \downarrow ((\alpha \downarrow \alpha) \downarrow (\beta \downarrow \beta)). \end{aligned}$$

5. Show, that

a) if $\Sigma \models \phi$ and $\Sigma \models \neg\phi$ for some ϕ, then Σ is unsatisfiable.

Solution. Assume that for some ϕ it holds $\Sigma \models \phi$ and $\Sigma \models \neg\phi$. We use **proof by contradiction**, that is, we assume that Σ is satisfiable and show that this leads to contradiction. If Σ is satisfiable then there is a truth assignment $\mathcal{A}$ such that for all $\sigma \in \Sigma$, $\mathcal{A} \models \sigma$. Since $\Sigma \models \phi$, it holds $\mathcal{A} \models \phi$. On the other hand, since $\Sigma \models \neg\phi$, it holds $\mathcal{A} \models \neg\phi$, which is equivalent to $\mathcal{A} \not\models \phi$. Since no proposition can be true and false at the same time, this is a contradiction and the original claim holds, that is, Σ is unsatisfiable. $\square$

b) if set of propositions Σ has exactly one model, then for all ϕ it holds $\Sigma \models \phi$ or $\Sigma \models \neg\phi$ (but not both).

Solution. Let $\mathcal{A}$ be the only model for Σ. For each proposition ϕ it holds that ϕ is **either** true in $\mathcal{A}$ **or** ϕ is false in $\mathcal{A}$, that is, either $\mathcal{A} \models \phi$ or $\mathcal{A} \not\models \phi$ (equivalently $\mathcal{A} \models \neg\phi$). If $\mathcal{A} \models \phi$, it holds $\Sigma \models \phi$, and if $\mathcal{A} \models \neg\phi$, it holds $\Sigma \models \neg\phi$. $\square$

6. Proof the following properties of logical consequences.

a) $\Sigma \subseteq \mathrm{Cn}(\Sigma)$.

b) Monotonicity: $\Sigma_1 \subseteq \Sigma_2 \Rightarrow \mathrm{Cn}(\Sigma_1) \subseteq \mathrm{Cn}(\Sigma_2)$.

c) $\Sigma \models \phi \Rightarrow \mathrm{Cn}(\Sigma) = \mathrm{Cn}(\Sigma \cup \{\phi\})$.

Solution. $\mathrm{Cn}(\Sigma)$ denotes the set of logical consequences of a set of propositions Σ, that is, $\mathrm{Cn}(\Sigma) = \{\phi \mid \Sigma \models \phi\}$.

a) Assume that $\Sigma \not\subseteq \mathrm{Cn}(\Sigma)$. Then Σ contains α such that there is a model $\mathcal{A}$ of Σ that is not a model of α, that is, $\mathcal{A} \not\models \alpha$. On the other hand, since $\mathcal{A}$ is a model of Σ it holds $\mathcal{A} \models \sigma$ for all $\sigma \in \Sigma$. Since $\alpha \in \Sigma$, we have $\mathcal{A} \models \alpha$. This is a contradiction, and thus $\Sigma \subseteq \mathrm{Cn}(\Sigma)$. $\square$

b) Consider arbitrary $\alpha \in \mathrm{Cn}(\Sigma_1)$. α is true in all the models of Σ_1, that is, in all the truth assignments in which all the propositions in Σ_1 are true. Because $\Sigma_1 \subseteq \Sigma_2$, every model of Σ_2 is also a model of Σ_1. This implies that α is true in every model of Σ_2, that is, $\alpha \in \mathrm{Cn}(\Sigma_2)$. $\square$

c) Assume that $\Sigma \models \phi$, that is, for all $\mathcal{A}$ such that $\mathcal{A} \models \sigma$ for all $\sigma \in \Sigma$, it holds $\mathcal{A} \models \phi$. Based on item b) it holds $\mathrm{Cn}(\Sigma) \subseteq \mathrm{Cn}(\Sigma \cup \{\phi\})$ and it suffices to show that $\mathrm{Cn}(\Sigma \cup \{\phi\}) \subseteq \mathrm{Cn}(\Sigma)$. Consider arbitrary $\alpha \in \mathrm{Cn}(\Sigma \cup \{\phi\})$. It holds $\Sigma \cup \{\phi\} \models \alpha$, that is, α is true in every model of $\Sigma \cup \{\phi\}$. But these are exactly the same as the models of Σ, that is, $\Sigma \models \alpha$ and $\alpha \in \mathrm{Cn}(\Sigma)$. $\square$

7. Model with propositional logic a voting system for three voters, where the models give the positive or negative voting result. How does the system change if there are four voters and the vote chair decides in case of a tie.

Solution. We choose the following atomical propositions.

$$\begin{aligned} A &= \text{“person 1 votes yes”} \\ B &= \text{“person 2 votes yes”} \\ C &= \text{“person 3 votes yes”} \\ Y &= \text{“majority of yes-votes”} \end{aligned}$$

Two yes-votes results in majority for yes.

$$A \wedge B \rightarrow Y \quad A \wedge C \rightarrow Y \quad B \wedge C \rightarrow Y$$

Two no-votes results in minority of yes votes.

$$\neg A \wedge \neg B \rightarrow \neg Y \quad \neg A \wedge \neg C \rightarrow \neg Y \quad \neg B \wedge \neg C \rightarrow \neg Y$$

When there are three persons and a chairperson, we take in addition the following atomical propositions.

$$\begin{aligned} P &= \text{“chair votes yes”} \\ IC &= \text{“result of the vote depends on the vote of the chair”} \end{aligned}$$

Three yes or no votes gives the result directly.

$$A \wedge B \wedge C \rightarrow Y \quad \neg A \wedge \neg B \wedge \neg C \rightarrow \neg Y$$

Otherwise, the vote of the chairperson impacts the outcome of the vote.

$$\begin{array}{ccc} A \wedge \neg B \wedge \neg C \rightarrow IC & \neg A \wedge B \wedge \neg C \rightarrow IC & \neg A \wedge \neg B \wedge C \rightarrow IC \\ A \wedge B \wedge \neg C \rightarrow IC & A \wedge \neg B \wedge C \rightarrow IC & \neg A \wedge B \wedge C \rightarrow IC \end{array}$$

The impact of the chairperson's vote.

$$IC \wedge P \rightarrow Y \quad IC \wedge \neg P \rightarrow \neg Y$$

Naturally, there are also several other possibilities how to model the voting system.

8. The card reader of the Helsinki area travel card work as follows.

1. Green light: a valid period ticket / value ticket / transfer ticket.
2. Green and yellow light: less than or equal to 3 full days period on the travel card / less than or equal to 5 euros value on the travel card.
3. Red light: period / transfer not valid or other error.

Formalize the system using propositional logic and find out what kind of models the set of propositions has.

Solution. Choose for example the following atomical propositions.

$$\begin{array}{lcl} A & = & \text{“a valid period ticket on the card”} \\ B & = & \text{“a valid value ticket on the card”} \\ C & = & \text{“a valid transfer ticket on the card”} \end{array} \qquad \begin{array}{lcl} D & = & \text{“period} \leq 3 \text{ days”} \\ E & = & \text{“value} \leq 5 \text{ euros”} \\ F & = & \text{“other error”} \end{array}$$

$$\begin{array}{lcl} V & = & \text{“green light in the reader”} \\ K & = & \text{“yellow light in the reader”} \\ P & = & \text{“red light in the reader”} \end{array}$$

Now the system can be formalized for example as follows.

1. $A \vee B \vee C \rightarrow V$
2. $D \vee E \rightarrow K \wedge V$
3. $\neg A \vee \neg C \vee F \rightarrow P$