T-79.3001 Logic in computer science: foundations **Spring 2007**
Exercise 2 (propositional logic 2.1 – 3.5)
January 30– February 1, 2007

Solutions to demonstration problems

4. Define the connectives in propositional logic

a) using the proposition that is always false ($\bot$) and implication ($\rightarrow$).
Solution.

$$
\begin{aligned}
&\neg A \equiv A \rightarrow \bot\\
&A \vee B = \neg A \rightarrow B \equiv (A \rightarrow \bot) \rightarrow B\\
&A \wedge B = \neg(\neg A \vee \neg B) = \neg(A \rightarrow \neg B) = \neg(A \rightarrow (B \rightarrow \bot)) \equiv\\
&(A \rightarrow (B \rightarrow \bot)) \rightarrow \bot\\
&A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A) \equiv\\
&((A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow \bot)) \rightarrow \bot
\end{aligned}
$$

b) using Sheffer stroke.
Solution. Sheffer stroke is defined as $A \mid B = \neg(A \wedge B)$.

$$
\begin{aligned}
&\neg A \equiv A \mid A\\
&A \wedge B = \neg(A \mid B) \equiv (A \mid B) \mid (A \mid B)\\
&A \vee B = \neg(\neg A \wedge \neg B) = (\neg A \mid \neg B) \equiv (A \mid A) \mid (B \mid B)\\
&A \rightarrow B = \neg A \vee B = \neg(A \wedge \neg B) = (A \mid \neg B) \equiv (A \mid (B \mid B))\\
&A \leftrightarrow B = A \rightarrow B \wedge B \rightarrow A = (A \mid (B \mid B)) \wedge (B \mid (A \mid A)) \equiv\\
&((A \mid (B \mid B)) \mid (B \mid (A \mid A))) \mid ((A \mid (B \mid B)) \mid (B \mid (A \mid A)))
\end{aligned}
$$

5. List all possible binary connectives (16 in total) and give their definitions using the basic connectives in propositional logic.

All possibilities are listed in the following table.

p_0	t	t	f	f
p_1	t	f	t	f
$p_0 \vee \neg p_0$	t	t	t	t
$p_0 \vee p_1$	t	t	t	f
$p_1 \rightarrow p_0$	t	t	f	t
p_0	t	t	f	f
$p_0 \rightarrow p_1$	t	f	t	t
p_1	t	f	t	f
$p_0 \leftrightarrow p_1$	t	f	f	t
$p_0 \wedge p_1$	t	f	f	f

p_0	t	t	f	f
p_1	t	f	t	f
$p_0 \mid p_1$	f	t	t	t
$\neg(p_0 \leftrightarrow p_1)$	f	t	t	f
$\neg p_1$	f	t	f	t
$\neg(p_0 \rightarrow p_1)$	f	t	f	f
$\neg p_0$	f	f	t	t
$\neg(p_1 \rightarrow p_0)$	f	f	t	f
$p_0 \downarrow p_1$	f	f	f	t
$p_0 \wedge \neg p_0$	f	f	f	f

6. Let $\mathcal{A}_1 \subseteq \mathcal{P}$ and $\mathcal{A}_2 \subseteq \mathcal{P}$ be truth assignments and $\phi \in \mathcal{L}$ a proposition. Show that if $\mathcal{A}_1 \cap At(\phi) = \mathcal{A}_2 \cap At(\phi)$, then $\mathcal{A}_1 \models \phi \iff \mathcal{A}_2 \models \phi$.

Solution. Proof by induction.

Basic case: Let ϕ be an atomic proposition, that is, $At(\phi) = \{\phi\}$. By the definition of intersection either $\phi \in \mathcal{A}_1$ and $\phi \in \mathcal{A}_2$, which implies $\mathcal{A}_1 \models \phi$ and $\mathcal{A}_2 \models \phi$, or $\phi \notin \mathcal{A}_1$ and $\phi \notin \mathcal{A}_2$, which implies $\mathcal{A}_1 \not\models \phi$ and $\mathcal{A}_2 \not\models \phi$. Thus $\mathcal{A}_1 \models \phi \iff \mathcal{A}_2 \models \phi$.

Induction hypothesis: The claim holds for all ϕ that have at most n connectives.

Induction step: Let ϕ a proposition that has $n+1$ connectives. Let's do a case analysis for different connectives.

1. Let ϕ be of the form $\neg\alpha$. Now, by induction hypothesis, the claim holds for proposition α. If $\mathcal{A}_1 \models \alpha$ and $\mathcal{A}_2 \models \alpha$, then $\mathcal{A}_1 \not\models \neg\alpha$ and $\mathcal{A}_2 \not\models \neg\alpha$. On the other hand, if $\mathcal{A}_1 \not\models \alpha$ and $\mathcal{A}_2 \not\models \alpha$, then $\mathcal{A}_1 \models \neg\alpha$ and $\mathcal{A}_2 \models \neg\alpha$. Thus the claim holds , if ϕ is of the form $\neg\alpha$.
2. Let ϕ be of the form $\alpha \wedge \beta$. The claim holds for both α and β by the induction hypothesis. There are four possible cases.
   - If $\mathcal{A}_1 \models \alpha$, $\mathcal{A}_2 \models \alpha$, $\mathcal{A}_1 \models \beta$ and $\mathcal{A}_2 \models \beta$, then it holds $\mathcal{A}_1 \models \alpha \wedge \beta$ and $\mathcal{A}_2 \models \alpha \wedge \beta$.
   - If $\mathcal{A}_1 \models \alpha$, $\mathcal{A}_2 \models \alpha$, $\mathcal{A}_1 \not\models \beta$ and $\mathcal{A}_2 \not\models \beta$, then it holds $\mathcal{A}_1 \not\models \alpha \wedge \beta$ and $\mathcal{A}_2 \not\models \alpha \wedge \beta$.
   - If $\mathcal{A}_1 \not\models \alpha$, $\mathcal{A}_2 \not\models \alpha$, $\mathcal{A}_1 \models \beta$ and $\mathcal{A}_2 \models \beta$, then it holds $\mathcal{A}_1 \not\models \alpha \wedge \beta$ and $\mathcal{A}_2 \not\models \alpha \wedge \beta$.
   - If $\mathcal{A}_1 \not\models \alpha$, $\mathcal{A}_2 \not\models \alpha$, $\mathcal{A}_1 \not\models \beta$ and $\mathcal{A}_2 \not\models \beta$, then it holds $\mathcal{A}_1 \not\models \alpha \wedge \beta$ and $\mathcal{A}_2 \not\models \alpha \wedge \beta$.

   Thus, the claim holds if ϕ is of the form $\alpha \wedge \beta$.
3. Go similarly through the other connectives based on their definitions.

7. Let $\mathcal{A} = \emptyset$ be a truth assignment. Find the truth value of

$$(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$$

by using a) the truth table and b) the recursive definition of truth values.

a) **Solution.** We denote the proposition with ϕ and choose the truth values for A and B according to $\mathcal{A}$.

A	B	$\neg A$	$\neg B$	$\neg B \rightarrow \neg A$	$\neg B \rightarrow A$	$(\neg B \rightarrow A) \rightarrow B$	ϕ
F	F	T	T	T	F	T	T

b) **Solution.**

* According to the definition $A \notin \mathcal{A}$ iff $\mathcal{A} \not\models A$. Similarly $B \notin \mathcal{A}$ iff $\mathcal{A} \not\models B$.
* Based on the definition of negation $\mathcal{A} \not\models A$ iff $\mathcal{A} \models \neg A$ and $\mathcal{A} \not\models B$ iff $\mathcal{A} \models \neg B$.
* Since $\mathcal{A} \models \neg A$, it holds $\mathcal{A} \models \neg B \rightarrow \neg A$.
* Since $\mathcal{A} \not\models A$ and $\mathcal{A} \models \neg B$, we have $\mathcal{A} \not\models \neg B \rightarrow A$.
* Because $\mathcal{A} \not\models \neg B \rightarrow A$, it holds $\mathcal{A} \models (\neg B \rightarrow A) \rightarrow B$.
* Since $\mathcal{A} \models (\neg B \rightarrow A) \rightarrow B$, we have $\mathcal{A} \models \phi$.

8. An engineer designed a specification for two traffic light posts positioned in the intersection of two one-way streets:

(i) Both lights have a green, a yellow and a red light. Exactly one of the lights is lit on both posts all times.

(ii) Both green lights are not on at the same time.

(iii) If one lamp post has the red light on, then the other has either green or yellow light on.

a) Formalize the above requirements as a set of propositional logic statements.

b) Construct a truth table for the set of statements.

c) Give a model for the set of statements and a truth assignment such that the set of statements is not satisfied.

d) Are the requirements complete enough for a real life situation?

Solution.

a) We will use atomic propositions $P1$, $K1$ and $V1$ to denote respectively that the lamp post 1 has red, yellow and green light on (the letters come from the initial letters of the colors in Finnish). Let $P2$, $K2$ and $V2$ be the corresponding propositions for lamp post 2. Now we'll go through each requirement and present the set of propositions that correspond to the requirement.

(i) For lamp post 1 we need proposition $P1 \vee K1 \vee V1$ (at least one lamp is alight) and propositions $P1 \rightarrow \neg K1 \wedge \neg V1$, $K1 \rightarrow \neg P1 \wedge \neg V1$, $V1 \rightarrow \neg P1 \wedge \neg K1$ (at most one lamp is alight). Also, corresponding propositions are needed for lamp post 2.

(ii) The needed proposition is $\neg(V1 \wedge V2)$.

(iii) We need propositions $P1 \rightarrow (K2 \vee V2)$ and $P2 \rightarrow (K1 \vee V1)$.

b) Let's construct a truth table for the above set of propositions. We'll use a shorthand notation α_i for propositions $(Pi \vee Ki \vee Vi) \wedge (Pi \rightarrow \neg Ki \wedge \neg Vi) \wedge (Ki \rightarrow \neg Pi \wedge \neg Vi) \wedge (Vi \rightarrow \neg Pi \wedge \neg Ki)$ (which means that the lamp post i has exactly one light on). The rows marked with stars are models of the set of propositions.

P1	K1	V1	P2	K2	V2	α_1	α_2	$\neg(V1 \wedge V2)$	$P1 \rightarrow (K2 \vee V2)$	$P2 \rightarrow (K1 \vee V1)$	
F	F	F	F	F	F	F	F	T	T	T	
F	F	F	F	F	T	F	T	T	T	T	
F	F	F	F	T	F	F	T	T	T	T	
F	F	F	F	T	T	F	F	T	T	T	
F	F	F	T	F	F	F	T	T	T	F	
F	F	F	T	F	T	F	F	T	T	F	
F	F	F	T	T	F	F	F	T	T	F	
F	F	F	T	T	T	F	F	T	T	F	
F	F	T	F	F	F	T	F	T	T	T	
F	F	T	F	F	T	T	T	F	T	T	
F	F	T	F	T	F	T	T	T	T	T	*
F	F	T	F	T	T	T	F	F	T	T	
F	F	T	T	F	F	T	T	T	T	T	*
F	F	T	T	F	T	T	F	F	T	T	
F	F	T	T	T	F	T	F	T	T	T	
F	F	T	T	T	T	T	F	F	T	T	
F	T	F	F	F	F	T	F	T	T	T	
F	T	F	F	F	T	T	T	T	T	T	*
F	T	F	F	T	F	T	T	T	T	T	*
F	T	F	F	T	T	T	F	T	T	T	
F	T	F	T	F	F	T	T	T	T	T	*
F	T	F	T	F	T	T	F	T	T	T	
F	T	F	T	T	F	T	F	T	T	T	
F	T	F	T	T	T	T	F	T	T	T	
F	T	T	F	F	F	F	F	T	T	T	
F	T	T	F	F	T	F	T	F	T	T	
F	T	T	F	T	F	F	T	T	T	T	
F	T	T	F	T	T	F	F	F	T	T	
F	T	T	T	F	F	F	T	T	T	T	
F	T	T	T	F	T	F	F	F	T	T	
F	T	T	T	T	F	F	F	T	T	T	
F	T	T	T	T	T	F	F	F	T	T	

P1	K1	V1	P2	K2	V2	α_1	α_2	$\neg(V1 \wedge V2)$	$P1 \rightarrow (K2 \vee V2)$	$P2 \rightarrow (K1 \vee V1)$	
T	F	F	F	F	F	*T*	*F*	*T*	*F*	*T*	
T	F	F	F	F	T	*T*	*T*	*T*	*T*	*T*	*
T	F	F	F	T	F	*T*	*T*	*T*	*T*	*T*	*
T	F	F	F	T	T	*T*	*F*	*T*	*T*	*T*	
T	F	F	T	F	F	*T*	*T*	*T*	*F*	*F*	
T	F	F	T	F	T	*T*	*F*	*T*	*T*	*F*	
T	F	F	T	T	F	*T*	*F*	*T*	*T*	*F*	
T	F	F	T	T	T	*T*	*F*	*T*	*T*	*F*	
T	F	T	F	F	F	*F*	*F*	*T*	*F*	*T*	
T	F	T	F	F	T	*F*	*T*	*F*	*T*	*T*	
T	F	T	F	T	F	*F*	*T*	*T*	*T*	*T*	
T	F	T	F	T	T	*F*	*F*	*F*	*T*	*T*	
T	F	T	T	F	F	*F*	*T*	*T*	*F*	*T*	
T	F	T	T	F	T	*F*	*F*	*F*	*T*	*T*	
T	F	T	T	T	F	*F*	*F*	*T*	*T*	*T*	
T	F	T	T	T	T	*F*	*F*	*F*	*T*	*T*	
T	T	F	F	F	F	*F*	*F*	*T*	*F*	*T*	
T	T	F	F	F	T	*F*	*T*	*T*	*T*	*T*	
T	T	F	F	T	F	*F*	*T*	*T*	*T*	*T*	
T	T	F	F	T	T	*F*	*F*	*T*	*T*	*T*	
T	T	F	T	F	F	*F*	*T*	*T*	*F*	*T*	
T	T	F	T	F	T	*F*	*F*	*T*	*T*	*T*	
T	T	F	T	T	F	*F*	*F*	*T*	*T*	*T*	
T	T	F	T	T	T	*F*	*F*	*T*	*T*	*T*	
T	T	T	F	F	F	*F*	*F*	*T*	*F*	*T*	
T	T	T	F	F	T	*F*	*T*	*F*	*T*	*T*	
T	T	T	F	T	F	*F*	*T*	*T*	*T*	*T*	
T	T	T	F	T	T	*F*	*F*	*F*	*T*	*T*	
T	T	T	T	F	F	*F*	*T*	*T*	*F*	*T*	
T	T	T	T	F	T	*F*	*F*	*F*	*T*	*T*	
T	T	T	T	T	F	*F*	*F*	*T*	*T*	*T*	
T	T	T	T	T	T	*F*	*F*	*F*	*T*	*T*	

There are seven models (out of $2^6 = 64$ valuations). The claim "both red lights are not on at the same time" can be formalized as $\neg(P1 \wedge P2)$. Examining the models we can see that the proposition $\neg(P1 \wedge P2)$ is true in each of them (check it), so it is a logical consequence of the set of propositions.

c) The claim "the yellow light is alight on both traffic lights" translates into proposition $K1 \wedge K2$. Let $\mathcal{A}_1$ be a truth assignment that maps $K1$ and $K2$ to true and all other atomic propositions to false, that is, $\mathcal{A}_1 = \{K1, K2\}$. Now, $\mathcal{A}_1 \models (K1 \wedge K2)$, since $\mathcal{A}_1 \models K1$ ja $\mathcal{A}_1 \models K2$). In addition $\mathcal{A}_1 \models \alpha$ holds for all propositions α in item (a) (check!). Thus $\mathcal{A}_1$ is a model of the set of propositions, where $K1 \wedge K2$ is true. Let $\mathcal{A}_2$ be a truth assignment that maps propositions $V1$ and $V2$ to true and all other atomic propositions to false, that is, $\mathcal{A}_2 = \{V1, V2\}$. Now $\mathcal{A}_2 \not\models \neg(V1 \wedge V2)$, and thus the set of propositions is not satisfied in $\mathcal{A}_2$.

d) The requirements are not sufficient, because in real life red and yellow lights may be on at the same time. It is possible to lighten the conditions of (i) to allow this (think how this may be done by yourself). A worse problem is that the propositions don't specify the working order of the lights (e.g. that the yellow light should follow the green one). It is quite difficult to model this kind of behaviour with propositional logic.

9. Apply truth tables to see whether the following claims hold.

a) $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ is valid.

b) $\neg((A \rightarrow B) \rightarrow ((\neg A \rightarrow B) \rightarrow B))$ is unsatisfiable.

c) $A \leftrightarrow B$ and $\neg(A \leftrightarrow \neg B)$ are logically equivalent.

d) $\{(A \wedge B) \vee (C \wedge A), (A \wedge B) \vee \neg B\} \models A \vee (C \wedge \neg B)$.

Solution.

a) Components of $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ are: $A, B, C, A \rightarrow B, A \rightarrow C, B \rightarrow C$, $(B \rightarrow C) \rightarrow (A \rightarrow C)$ and itself (we denote it by ϕ). Proposition ϕ is valid iff ϕ is true in all possible truth assignments.

A	B	C	$A \rightarrow B$	$A \rightarrow C$	$B \rightarrow C$	$(B \rightarrow C) \rightarrow (A \rightarrow C)$	ϕ
T	*T*	*T*	*T*	*T*	*T*	*T*	*T*
T	*T*	*F*	*T*	*F*	*F*	*T*	*T*
T	*F*	*T*	*F*	*T*	*T*	*T*	*T*
T	*F*	*F*	*F*	*F*	*T*	*F*	*T*
F	*T*	*T*	*T*	*T*	*T*	*T*	*T*
F	*T*	*F*	*T*	*T*	*F*	*T*	*T*
F	*F*	*T*	*T*	*T*	*T*	*T*	*T*
F	*F*	*F*	*T*	*T*	*T*	*T*	*T*

The last column only contains T and thus ϕ is valid.

b) The proposition is unsatisfiable iff all the values in the column of the truth table corresponding to it are F.

c)

A	B	$A \leftrightarrow B$	$\neg A \leftrightarrow \neg B$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Since the columns for $A \leftrightarrow B$ and $\neg A \leftrightarrow \neg B$ are identical, the propositions are logically equivalent.

d)

A	B	C	$(A \wedge B) \vee (C \wedge A)$	$(A \wedge B) \vee \neg B$	$A \vee (C \wedge \neg B)$
T	T	T	T	T	$T\star$
T	T	F	T	T	$T\star$
T	F	T	T	T	$T\star$
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	T	T
F	F	F	F	T	F

The claim holds, because $A \vee (C \wedge \neg B)$ has the value T in all the lines in which $(A \wedge B) \vee (C \wedge A)$ and $(A \wedge B) \vee \neg B$ get the value T (marked with $\star$).