T-79.3001 Logic in computer science: foundations Spring 2007 Exercise 10 ([NS, 1997], Predicate Logic, Chapters 6 – 7)
April 17–19, 2007

Solutions to demonstration problems

- **4.** A *directed* graph consists of a set of nodels and a set of *directed* egdes between the nodes. Assume that nodes a represented with constants $\{a, b, \ldots\}$ and edges with a binary predicate K(x, y) = "there is an edge from node x to node y".
 - 1. Define predicates $R_n(x,y) =$ "node y is reachable from node x using n edges", for n = 0, 1, 2, ..., k. Represent the following graph with predicate K.

$$a \stackrel{\longrightarrow}{\sqsubseteq} b \longrightarrow c$$

2. Use semantic tableaux to show that

$$\exists x (R_2(x,x) \land R_3(x,c))$$

is a logical consequence of the representation of the graph and definitions of predicates R_2 and R_3

Solution.

a) Define predicates $R_n(x, y)$ as follows:

$$\forall x R_0(x, x)$$

$$\forall x \forall y \forall z (R_0(x, y) \land K(y, z) \rightarrow R_1(x, z))$$

$$\forall x \forall y \forall z (R_1(x, y) \land K(y, z) \rightarrow R_2(x, z))$$

$$\vdots$$

$$\forall x \forall y \forall z (R_{k-1}(x, y) \land K(y, z) \rightarrow R_k(x, z))$$

The graph

$$a \xrightarrow{\longrightarrow} b \longrightarrow c$$

can be represented as K(a,b), K(b,a) and K(b,c).

b)
$$1. T(\forall x R_0(x,x))$$

$$2. T(\forall x \forall y \forall z (R_0(x,y) \land K(y,z) \rightarrow R_1(x,z)))$$

$$3. T(\forall x \forall y \forall z (R_1(x,y) \land K(y,z) \rightarrow R_2(x,z)))$$

$$4. T(\forall x \forall y \forall z (R_2(x,y) \land K(y,z) \rightarrow R_3(x,z)))$$

$$5. T(K(a,b))$$

$$6. T(K(b,a))$$

$$7. T(K(b,c))$$

$$8. F(\exists x (R_2(x,x) \land R_3(x,c)))$$

$$9. F(R_2(b,b) \land R_3(b,c)), 8. x/b$$

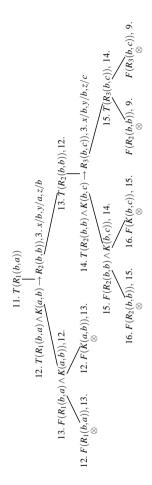
$$10. T(R_0(b,b) \land K(b,a) \rightarrow R_1(b,a)), 2. x/b, y/b, z/a$$

$$11. F(R_0(b,b) \land K(b,a)), 10. \quad 11. T(R_1(b,a)), 10.$$

$$12. F(R_0(b,b)), 11. \quad 12. F(K(b,a)), 11.$$

$$13. T(R_0(b,b)), 1.$$

The subtree from node 11 continues in the next page.



The finished tableau is contradictory and the claim holds.

5. We represent binary trees using a binary function s (internal nodes) and a unary function l (leaf nodes). Thus the representation of the upper tree is the term s(s(l(c), l(a)), l(b)).



- a) Let predicate PK(x,y) denote that binary tree x is the mirror-image of binary tree y. Define predicate PK.
- b) Use semantic tableaux to proof that the upper binary tree is the mirror-image of the lower binary tree.

Solution. Define predicate PK as follows:

$$\forall x PK(l(x), l(x))$$

$$\forall x \forall y \forall v \forall w (PK(x, v) \land PK(y, w) \rightarrow PK(s(x, y), s(w, v)))$$

We show that

$$PK(s(s(l(c), l(a)), l(b)), s(l(b), s(l(a), l(c))))$$

is a logical consequence of the definition of PK with the semantic tableau given in the next page.

$$T\forall x P K(l(x), l(x))$$

$$T\forall x \forall y \forall z \forall v (P K(x, y) \land P K(z, v) \rightarrow P K(s(x, z), s(v, y)))$$

$$FP K(s(s(l(c), l(a)), l(b)), s(l(b), s(l(a), l(c))))$$

$$TP K(l(a), l(a))$$

$$TP K(l(a), l(a))$$

$$TP K(l(b), l(b))$$

$$TP K(l(c), l(c))$$

$$T\forall y \forall y \forall v (P K(s(l(c), l(a)), y) \land P K(z, v) \rightarrow P K(s(s(l(c), l(a)), z), s(v, y)))$$

$$T\forall v (P K(s(l(c), l(a)), s(l(a), l(c))) \land P K(z, v) \rightarrow P K(s(s(l(c), l(a)), z), s(v, s(l(a), l(c)))))$$

$$T\forall v (P K(s(l(c), l(a)), s(l(a), l(c))) \land P K(l(b), v) \rightarrow P K(s(s(l(c), l(a)), l(b)), s(v, s(l(a), l(c)))))$$

$$T(P K(s(l(c), l(a)), s(l(a), l(c))) \land P K(l(b), l(b)) \rightarrow P K(s(s(l(c), l(a)), l(b)), s(l(a), l(c))))$$

$$F(P K(s(l(c), l(a)), s(l(a), l(c))) \land P K(l(b), l(b)) \rightarrow P K(s(s(l(c), l(a)), l(b)), s(l(a), l(c))))$$

$$FP K(s(l(c), l(a)), s(l(a), l(c))) \land P K(l(b), l(b))$$

$$T\forall y \forall z \forall v (P K(l(c), v) \land P K(z, v) \rightarrow P K(s(l(c), z), s(v, l(c))))$$

$$T\forall v (P K(l(c), l(c)) \land P K(l(a), v) \rightarrow P K(s(l(c), l(a)), s(v, l(c))))$$

$$T(P K(l(c), l(c)) \land P K(l(a), l(a)) \rightarrow P K(s(l(c), l(a)), s(l(a), l(c)))$$

$$F(P K(l(c), l(c)) \land P K(l(a), l(a)) \rightarrow P K(s(l(c), l(a)), s(l(a), l(c)))$$

$$F(P K(l(c), l(c)) \land P K(l(a), l(a)) \rightarrow P K(s(l(c), l(a)), s(l(a), l(c)))$$

$$F(P K(l(c), l(c)) \land P K(l(a), l(a)) \rightarrow P K(s(l(c), l(a)), s(l(a), l(c)))$$

$$F(P K(l(c), l(c)) \land P K(l(a), l(a)) \rightarrow P K(s(l(c), l(a)), s(l(a), l(c)))$$

$$F(P K(l(c), l(c)) \land P K(l(a), l(a)) \rightarrow P K(s(l(c), l(a)), s(l(a), l(c)))$$

$$F(P K(l(c), l(c)) \land P K(l(a), l(a)) \rightarrow P K(s(l(c), l(a)), s(l(a), l(c)))$$

$$F(P K(l(c), l(c)) \land P K(l(a), l(a)) \rightarrow P K(s(l(c), l(a)), s(l(a), l(c)))$$

6. Quantifier $\exists !x$ is used to denote "there is only one x". Sentence $\exists !x \phi(x)$ can be represented as

$$(\exists x \phi(x)) \land (\forall x \forall y (\phi(x) \land \phi(y) \rightarrow x = y)).$$

Formalize the following sentences using predicate logic:

- There is only one Father Christmas.
- Every Santa Claus is Father Christmas
- Every Father Christmas is Santa Claus.
- There is only one Santa Clause.

Use semantic tableaux to prove that sentence 4 is a logical consequence of

J(x) denote that x is Santa Claus. Thus we get the following sentences: **Solution.** Let predicate K(x) denote that x is Father Christmas and predicate 1. $\exists x K(x) \land \forall x \forall y (K(x) \land K(y) \rightarrow x = y),$

Sentence 4 is of the form: $\exists x J(x) \land \forall x \forall y (J(x) \land J(y) \rightarrow x = y)$. 3. $\forall x(K(x) \rightarrow J(x))$.

2. $\forall x(J(x) \rightarrow K(x))$, ja

The semantic tableau:

$$1. T(\exists xK(x) \land \forall x \forall y(K(x) \land K(y) \rightarrow x = y))$$

$$2. T(\forall x(J(x) \rightarrow K(x)))$$

$$3. T(\forall x(K(x) \rightarrow J(x)))$$

$$4. F(\exists xJ(x) \land \forall x \forall y(J(x) \land J(y) \rightarrow x = y))$$

$$5. T(\exists xK(x)), 1.$$

$$6. T(\forall x \forall y(K(x) \land K(y) \rightarrow x = y))$$

$$7. F(\exists xJ(x)) \qquad 7. F(\forall x \forall y(J(x) \land J(y) \rightarrow x = y))$$

$$8. T(K(a)), 5. x/a \qquad 8. F(\forall y(J(b) \land J(y) \rightarrow b = y)), 7. x/b$$

$$9. F(J(a)), 7. x/a \qquad 9. F(J(b) \land J(c) \rightarrow b = c), 8. y/c$$

$$10. T(K(a) \rightarrow J(a)), 3. \qquad 10. T(J(b) \land J(c))$$

$$11. F(K(a)) \qquad 11. T(J(a)) \qquad 11. F(b = c)$$

$$12. T(J(b))$$

$$13. T(J(c))$$

$$14. T(K(b) \land K(c) \rightarrow b = c), 6. x/b, y/c$$

$$15. F(K(b) \land K(c)) \qquad 15. T(b = c)$$

$$16. F(K(c)) \qquad 16. F(K(c)$$

$$17. T(J(b) \rightarrow K(b)), 2. x/b \qquad 17. T(J(c) \rightarrow K(c)), 3. x/c$$

$$18. F(J(b)) \qquad 18. T(K(b)) \qquad 18. F(J(c)) \qquad 18. T(K(c))$$