T-79.3001 Logic in computer science: foundations **Spring 2007**
Exercise 10 ([NS, 1997], Predicate Logic, Chapters 6 – 7)
April 17–19, 2007

Solutions to demonstration problems

4. A *directed* graph consists of a set of nodels and a set of *directed* egdes between the nodes. Assume that nodes a represented with constants $\{a,b,\ldots\}$ and edges with a binary predicate $K(x,y)$ = “there is an edge from node x to node y“.

1. Define predicates $R_n(x,y) =$ “node y is reachable from node x using n edges”, for $n = 0,1,2,\ldots,k$. Represent the following graph with predicate K.

$$\begin{tikzcd} a \arrow[r, shift left] & b \arrow[l, shift left] \arrow[r] & c \end{tikzcd}$$

2. Use semantic tableaux to show that

$$\exists x(R_2(x,x) \land R_3(x,c))$$

is a logical consequence of the representation of the graph and definitions of predicates R_2 and R_3

Solution.

a) Define predicates $R_n(x,y)$ as follows:

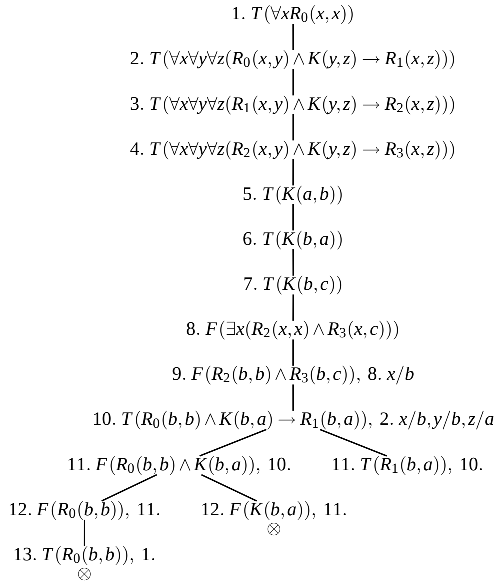
$$\forall x R_0(x,x)$$
$$\forall x\forall y\forall z(R_0(x,y) \land K(y,z) \to R_1(x,z))$$
$$\forall x\forall y\forall z(R_1(x,y) \land K(y,z) \to R_2(x,z))$$
$$\vdots$$
$$\forall x\forall y\forall z(R_{k-1}(x,y) \land K(y,z) \to R_k(x,z))$$

The graph

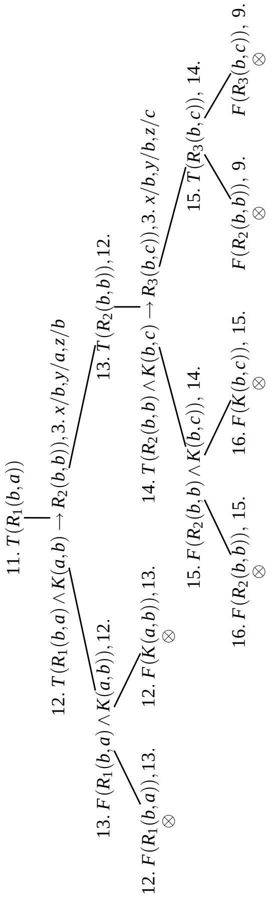
$$\begin{tikzcd} a \arrow[r, shift left] & b \arrow[l, shift left] \arrow[r] & c \end{tikzcd}$$

can be represented as $K(a,b)$, $K(b,a)$ and $K(b,c)$.

b)

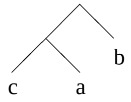


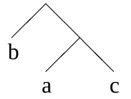
The subtree from node 11 continues in the next page.



The finished tableau is contradictory and the claim holds.

5. We represent binary trees using a binary function s (internal nodes) and a unary function l (leaf nodes). Thus the representation of the upper tree is the term $s(s(l(c),l(a)),l(b))$.





a) Let predicate $\mathrm{PK}(x,y)$ denote that binary tree x is the mirror-image of binary tree y. Define predicate PK.

b) Use semantic tableaux to proof that the upper binary tree is the mirror-image of the lower binary tree.

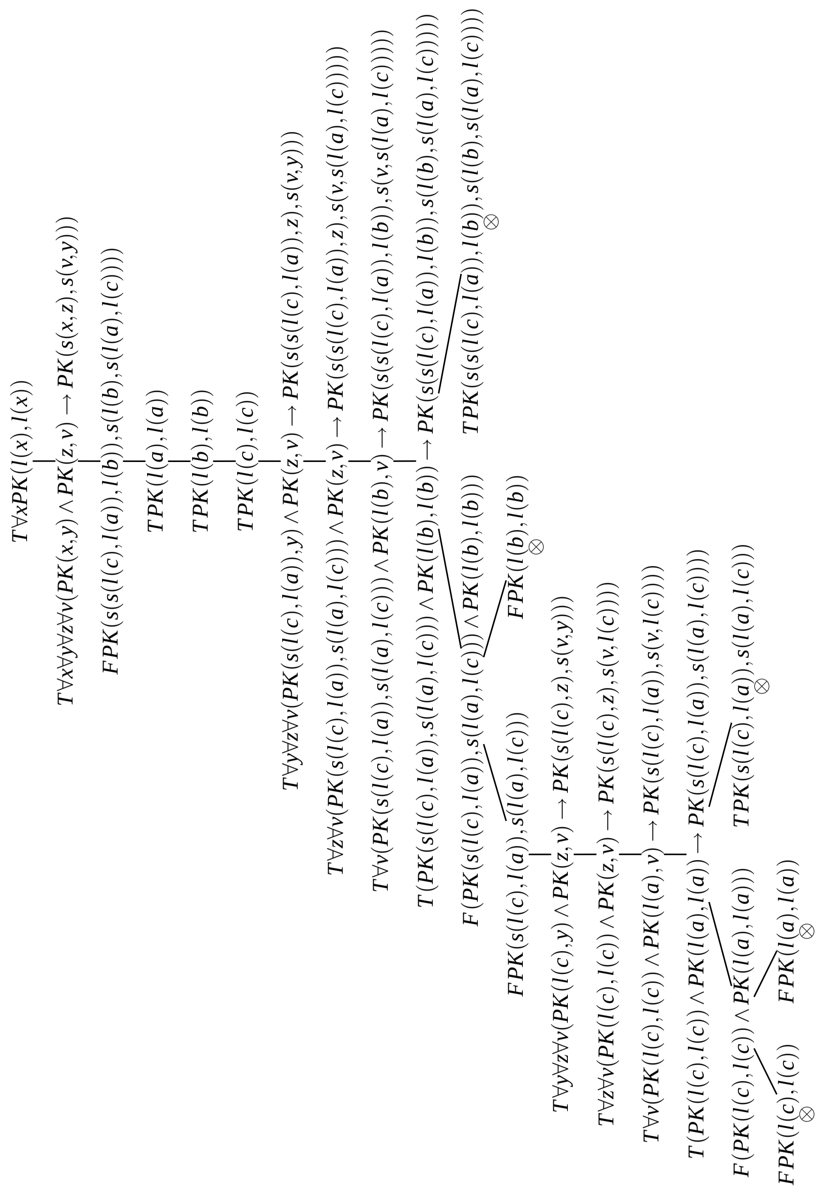
Solution. Define predicate PK as follows:

$$\forall x PK(l(x),l(x))$$
$$\forall x \forall y \forall v \forall w (PK(x,v) \land PK(y,w) \rightarrow PK(s(x,y),s(w,v)))$$

We show that

$$PK(s(s(l(c),l(a)),l(b)),s(l(b),s(l(a),l(c))))$$

is a logical consequence of the definition of PK with the semantic tableau given in the next page.



6. Quantifier $\exists! x$ is used to denote “there is only one x”. Sentence $\exists! x \phi(x)$ can be represented as

$$(\exists x \phi(x)) \land (\forall x \forall y (\phi(x) \land \phi(y) \to x = y)).$$

Formalize the following sentences using predicate logic:

1. There is only one Father Christmas.
2. Every Santa Claus is Father Christmas.
3. Every Father Christmas is Santa Claus.
4. There is only one Santa Clause.

Use semantic tableaux to prove that sentence 4 is a logical consequence of sentences 1-3.

Solution. Let predicate $K(x)$ denote that x is Father Christmas and predicate $J(x)$ denote that x is Santa Claus. Thus we get the following sentences:

1. $\exists x K(x) \land \forall x \forall y (K(x) \land K(y) \to x = y)$,
2. $\forall x (J(x) \to K(x))$, ja
3. $\forall x (K(x) \to J(x))$.

Sentence 4 is of the form: $\exists x J(x) \land \forall x \forall y (J(x) \land J(y) \to x = y)$.

The semantic tableau: