T-79.3001 Logic in computer science: foundations **Spring 2007**
Exercise 1 (preliminaries 3.1 – 3.2, propositional logic 1.1 – 1.5)
January 23–25, 2007

Solutions to demonstration problems

4. Show by induction that a set of n elements has 2^n subsets.

Basic case: A set of 0 elements (assuming $0 \in \mathbb{N}$), that is, the empty set, has one subset, itself. In addition $2^0 = 1$.

Induction hypothesis: We assume that the claim holds for $n = k$, that is, a set of k elements has 2^k subsets.

Induction step: Consider an arbitrary set A that has $k+1$ elements. Choose arbitrary $a \in A$. The subsets of A divide into two cases: the ones that contain a and the ones that don't contain a. Let B be the set of subsets of A containing a and C be the set of subsets of A not containing a. The set C is now the set of subsets of a k element set (consider why!) and by induction hypothesis $|C| = 2^k$. On the other hand, each element in B (that is, a each subset of A containing a) can be bijectively mapped into an element of C by removing a. Thus $|B| = |C|$. Since each subset of A belongs to either B or C (but not both!) the number of subsets of A is $|B| + |C| = 2^n + 2^n = 2 * 2^n = 2^{n+1}$. $\square$

5. Prove the following claims (sets A, B and C are subsets of universe E):

a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

b) $E - (A \cap B) = (E - A) \cup (E - B)$.

6. Formalize the following statements using propositional logic:

a) I can't finish my work unless you help me.

b) I either walk, ride a bicycle, or sometimes drive a car to work.

c) Merja and Arto are coming to visit us.

d) Because you have been naughty you won't get dessert.

e) Even though the manual was long it was not long enough.

f) If somebody asks me — or even if anybody doesn't ask — he shouldn't buy a car or he must live far from his workplace and the benzine should become cheaper.

Solution.

a) $\neg A \rightarrow \neg B$, when
A = "You help"
B = "I can finish my work"

b) $A \vee B \vee C$, when
A = "I walk to work"
B = "I ride a bicycle to work"
C = "Sometimes I drive a car to work"

c) Either: A, when
A = "Merja and Arto are coming to visit us"
or: $A \wedge B$, when
A = "Merja is coming to visit us"
B = "Arto is coming to visit us"

d) For example: $A \rightarrow \neg B$ or $A \wedge \neg B$, when
A = "You have been naughty"
B = "You will get dessert"

e) For example: $A \wedge B$, when
A = "The manual was long"
B = "The manual was not long enough"

f) $A \vee \neg A \rightarrow \neg B \vee (C \wedge D)$, when
A = "Somebody asks me"
B = "He should buy a car"
C = "He should live far from his workplace"
D = "The price of benzine should get cheaper"

7. Let $\mathcal{P} = \{A, B, C\}$ be the set of atomic statements. Which of the following are propositional statements? Why?

a) A

b) $\neg(A \wedge B))$

c) $(A \wedge (B \rightarrow (A \wedge C)))$

d) It is raining today.

Solution.

a) Yes, an atomic proposition.

b) No, does not contain even number of parentheses.

c) Yes, give for example a parse tree for the proposition.

d) No, natural language.

8. Prove that all propositional statements have an even number of parenthesis.

Solution. We prove the claim using induction on the length of proposition

Basic case: A proposition of the length 1 is a propositional symbol and it contains 0 parentheses (0 is an even number).

Induction hypothesis: If the length of the proposition is less than n, then it contains an even number of parenthesis.

Induction step: Let the length of the proposition to be n ($n > 1$). Then the proposition is of the form $(\neg\alpha)$, $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$ or $(\alpha \leftrightarrow \beta)$. Here α and β are some propositions that are shorter than n. By induction hypothesis they contain an even number of parenthesis and so the proposition itself contains also an even number of parenthesis. This completes the proof by induction.

9. Remove unnecessary parenthesis so that the meaning of the proposition does not change.

a) $(A \rightarrow ((B \wedge C) \vee D))$
b) $(((A \rightarrow B) \wedge (B \rightarrow C)) \rightarrow (A \rightarrow C))$
c) $((A \wedge (B \vee C)) \vee (A \wedge (C \vee D)))$
d) $((\neg(A \wedge B)) \leftrightarrow ((B \rightarrow C) \wedge A))$
e) $(((\neg A) \wedge (\neg B)) \rightarrow \neg(A \vee B))$

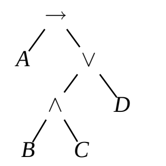
Solution. Using the common definitions for connective precedence:

a) $A \rightarrow (B \wedge C) \vee D$
b) $(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$
c) $(A \wedge (B \vee C)) \vee (A \wedge (C \vee D))$
d) $\neg(A \wedge B) \leftrightarrow (B \rightarrow C) \wedge A$
e) $\neg A \wedge \neg B \rightarrow \neg(A \vee B)$

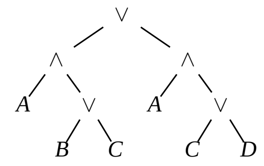
10. What are the forms of the propositional statements in the previous exercise? Give their parse trees.

Solution. The form of a statement is obtained from its outermost connective:

a) Implication.



b) Implication.
c) Disjunction.



d) Equivalence.
e) Implication.

11. List the components of the following propositional statement.

$$(\neg A \rightarrow (\neg B \rightarrow C)) \rightarrow (\neg(\neg A \rightarrow B) \rightarrow C)$$

Solution. $\neg A \rightarrow (\neg B \rightarrow C)$, $\neg(\neg A \rightarrow B) \rightarrow C$, $\neg A$, $\neg B \rightarrow C$, $\neg(\neg A \rightarrow B)$, C, A, $\neg B$, $\neg A \rightarrow B$, B. In addition the propositional statement is its own component.