Tutorial problems

1. Define predicate \( Y(x,y) \) (there is a connection from city \( x \) to city \( y \)) using the predicate \( L(x,y) \) (there is a flight from city \( x \) to city \( y \)).

2. Show that the following sentences are not valid by constructing a structure in which the sentence is false, i.e., construct a counter-example.
   a) \( \forall x (P(x) \rightarrow R(x)) \land \forall x (Q(x) \rightarrow R(x)) \rightarrow \forall x (P(x) \rightarrow Q(x)) \)
   b) \( \forall x \forall y (R(x,y) \rightarrow R(y,x)) \rightarrow \forall x R(a,x) \)

3. Transform the following sentences into clausal form.
   a) \( \neg(\exists x A(x) \lor \exists x B(x) \rightarrow \exists x (A(x) \lor B(x))) \)
   b) \( \neg(\forall x P(x) \rightarrow \exists x \forall y Q(x,y)) \lor \neg \forall y P(y) \)

Demonstration problems

4. Let \( R \) be a binary predicate with interpretation \( R^S \subseteq U \times U \) (the set \( U \) is the domain of structure \( S \)). In the following table we give definitions for some properties of relation \( R^S \).

<table>
<thead>
<tr>
<th>Property</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>reflexivity</td>
<td>( \forall x R(x,x) )</td>
</tr>
<tr>
<td>irreflexivity</td>
<td>( \forall x \neg R(x,x) )</td>
</tr>
<tr>
<td>symmetry</td>
<td>( \forall x \forall y (R(x,y) \rightarrow R(y,x)) )</td>
</tr>
<tr>
<td>asymmetry</td>
<td>( \forall x \forall y (R(x,y) \rightarrow \neg R(y,x)) )</td>
</tr>
<tr>
<td>transitivity</td>
<td>( \forall x \forall y \forall z (R(x,y) \land R(y,z) \rightarrow R(x,z)) )</td>
</tr>
<tr>
<td>seriality</td>
<td>( \forall x \exists y R(x,y) )</td>
</tr>
</tbody>
</table>

Consider a domain \( U \) consisting of people. Give examples of relations \( R^S \), \( (\emptyset \subset R^S \subset U^2) \), that have properties described above.

5. Show that the following sentences are not valid by constructing a structure in which the sentence is false, i.e., construct a counter-example.
   a) \( \forall x \exists y P(x,y) \rightarrow \exists y \forall x P(x,y) \)
6. Transform the following sentences into conjunctive normal form and perform skolemization.

a) \( \forall y (\exists x P(x,y) \to \forall z Q(y,z)) \land \exists y (\forall x R(x,y) \lor \forall x Q(x,y)) \)

b) \( \exists x \forall y R(x,y) \leftrightarrow \forall y \exists x P(x,y) \)

c) \( \forall x \exists y Q(x,y) \lor (\exists x \forall y P(x,y) \land \neg \exists x \exists y P(x,y)) \)

d) \( \neg (\forall x \exists y P(x,y) \to \exists x \exists y R(x,y)) \land \forall x \neg \exists y Q(x,y) \)

7. Use the rules in Lemma 9.1 [NS, 1997, page 129] to obtain rules for the following cases.

a) \( \forall x \phi(x) \to \psi \)

b) \( \exists x \phi(x) \to \psi \)

c) \( \phi \to \forall x \psi(x) \)

d) \( \phi \to \exists x \psi(x) \)

8. Transform the following sentences into clausal form.

a) \( \neg \exists x ((P(x) \to P(a)) \land (P(x) \to P(b))) \)

b) \( \forall y \exists x P(x,y) \)

c) \( \neg \forall y \exists x G(x,y) \)

d) \( \exists x \forall y \exists z (P(x,z) \lor P(z,y) \to G(x,y)) \)