EIGENVECTORS AND SPECTRA OF CAYLEY GRAPHS*

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1. Introduction

This manuscript illustrates from first principles how linear representation theory of finite groups enables the computation of the spectrum and an eigenvector basis for *Cayley graphs* [defined in Section 4]. Applications for such methodology include

- the analysis of random walks on Cayley graphs [1, 2, 10]; and
- algebraic graph theory: the spectrum of the adjacency matrix of a regular graph is related to several important graph-theoretic quantities such as connectivity and expansion [5, 10]; and
- the study of combinatorial landscapes [11] using fast Fourier transform techniques [12].

The organization of this manuscript is as follows: Section 2 gives a brief introduction to the fundamental concepts of linear representation theory. The irreducible linear representations for some specific groups are given as examples in Section 3. Cayley graphs are defined with examples in Section 4. Section 5 surveys the results of Diaconis and Shahshahani [1] and Rockmore *et al.* [12] that connect the inequivalent irreducible linear representations of a group to the eigenvalues and eigenvectors of a Cayley graph derived from the group. The spectra of a few selected Cayley graphs is computed as an example with the help of the representations given in Section 3.

2. Linear representation theory

This section follows Part I of Serre's textbook [15] to a large extent. More recent textbooks are [3] and [8]. For preliminaries in linear algebra and group theory see e.g. [6, 14].

2.1. **Assumptions and notation.** All vector spaces are over the complex field $\mathbb{C}$ and have finite dimension. For a finite group G we denote by $\mathbb{C}[G]$ the complex vector space of dimension $|G|$ with base $\{e_g\}_{g\in G}$. Thus, an element $\phi \in \mathbb{C}[G]$ has the form

$$\phi = \sum_{g\in G} \mu_g e_g, \qquad \mu_g \in \mathbb{C}.$$

We identify $\mathbb{C}[G]$ with the vector space of all complex-valued functions on G. Namely, a function $\phi : G \to \mathbb{C}$ corresponds to the vector $\phi = \sum_{g\in G} \phi(g)e_g$ and vice versa. In particular, the vectors $\{e_g\}_{g\in G}$ of the standard base correspond to the functions

$$e_g(h) = \begin{cases} 1 & \text{if } g = h\text{; and} \\ 0 & \text{otherwise.} \end{cases}$$

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The inner product of two vectors $\phi, \psi \in \mathbb{C}[G]$ is defined by

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi_g \bar{\psi}_g, \tag{1}$$

where $\bar{\psi}_g$ denotes the complex conjugate of ψ_g.

We use the symbol δ for the Kronecker delta, that is,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j\text{; and} \\ 0 & \text{if } i \neq j. \end{cases}$$

2.2. **Linear representations.** Let G be a finite group, and let V be a finite-dimensional complex vector space. A *linear representation* of G on V is a group homomorphism $\rho : G \to \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ denotes the group of bijective linear transformations on V. The *degree* of a representation is the dimension of V.

Example 2.1. Let G be a finite group. The *(left) regular representation* ρ_{reg} of G on $\mathbb{C}[G]$ is defined by its action on the base $\{e_h\}_{h \in G}$: for all $g, h \in G$

$$\rho_{\mathrm{reg}}(g) e_h = e_{gh}.$$

The regular representation has degree $|G|$.

Two representations $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ are *equivalent* if there exists a bijective linear map $T : V_1 \to V_2$ such that $\rho_2(g) = T \rho_1(g) T^{-1}$ for all $g \in G$.

2.3. **Matrix representations.** Let $\rho : G \to V$ be a degree d linear representation of G and let $B = \{b_1, \ldots, b_d\}$ be a base for V. The *matrix representation* of ρ relative to B is a map ϱ that associates to each $g \in G$ the $d \times d$ matrix $\varrho(g)$ of $\rho(g)$ relative to B. For $1 \leq i, j \leq d$, we write $\varrho_{ij}(g)$ for the row-i, column-j coefficient of $\varrho(g)$. In other words, the $\varrho_{ij}(g)$ are the unique complex numbers that satisfy

$$\rho(g) b_j = \sum_{i=1}^{d} \varrho_{ij}(g) b_i$$

for all $1 \leq i, j \leq d$.

A matrix representation ϱ defines a collection of d^2 vectors in $\mathbb{C}[G]$. Namely, for all $1 \leq i, j \leq d$

$$\varrho_{ij} = \sum_{g \in G} \varrho_{ij}(g) e_g$$

is a vector in $\mathbb{C}[G]$.

2.4. **Subrepresentations, irreducibility.** Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation and let W be a vector subspace of V. The subspace W is *ρ-invariant* if for all $g \in G$ and $w \in W$ it holds that $\rho(g) w \in W$.

If W is a ρ-invariant subspace of V, then the restriction $\rho|_W$ of ρ to W is a representation of G on W. Such a representation is said to be a *subrepresenation* of ρ.

A representation $\rho : G \to \mathrm{GL}(V)$ is said to be *irreducible* if it has no ρ-invariant subspaces other than the trivial subspaces V and 0. Otherwise a representation is said to be *reducible*.

Recall from linear algebra that a vector space V is the *direct sum* of vector subspaces W and W', denoted $V = W \oplus W'$ if every $v \in V$ decomposes uniquely to a sum $v = w + w'$ with $w \in W$ and $w' \in W'$. The subspace W' is called the *complement* of W in the decomposition $V = W \oplus W'$. Recall further that the complements of W (and thus the decompositions of V into a direct sum involving W) are in a bijective correspondence with projections onto W. Namely, a *projection* P of V onto W associated with the decomposition $V = W \oplus W'$ is the linear map defined by $Pv = w$ for all $v \in W$. Conversely, the kernel of a linear map P with image W and $Pw = w$ for all $w \in W$ is a complement of W in V.

Theorem 2.2 (Maschke's Theorem). *Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a finite group G on a finite-dimensional complex vector space V and let W be a ρ-invariant subspace of V. Then, there exists a ρ-invariant complement W' of W.*

Proof. Let P be the projection associated with any complement of W in V. Define a linear map P' by setting

$$P' = \frac{1}{|G|} \sum_{g \in G} \rho(g) P \rho(g)^{-1}.$$

Since W is ρ-invariant and $Pw = w$ for all $w \in W$, we have that $P'w = w$ for all $w \in W$. Moreover, W must be the image of P' since W is the image of P and both $\rho(g)$ and $\rho(g)^{-1}$ map W onto W for all $g \in G$. Thus, P' is a projection onto W that corresponds to some complement W' of W. Let now $w' \in W'$. It suffices to show that $P'\rho(g)w' = 0$ for all $g \in G$ to establish that W' is ρ-invariant. For each $g \in G$ we have

$$\rho(g) P' \rho(g)^{-1} = \frac{1}{|G|} \sum_{h \in G} \rho(gh) P \rho(gh)^{-1} = P'.$$

Thus, $P'\rho(g)w' = \rho(g)P'w' = \rho(g)0 = 0$. □

The above theorem states that a reducible representation decomposes into a direct sum of subrepresentations acting on ρ-invariant subspaces of V. Namely, let W be a ρ-invariant subspace of V. Then, the above theorem states that there exists a ρ-invariant complement W' of W. Thus $\rho(g)v = \rho(g)(w+w') = \rho(g)w + \rho(g)w' = \rho(g)|_W w + \rho(g)|_{W'} w'$ for all $v \in V$ and hence ρ is simply the direct sum of two subrepresentations $\rho|_W$ and $\rho|_{W'}$ acting on W and W'. We indicate this by writing $\rho = \rho|_W \oplus \rho|_{W'}$.

2.5. **Characters.** Let $A : V \to V$ be a linear map with a matrix (a_{ij}). Recall from linear algebra that the *trace* of A is the scalar

$$\mathrm{Tr}\, A = \sum_i a_{ii},$$

which is independent of the choice of base for V relative to which the matrix (a_{ij}) is constructed.

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G on V. The *character* $\chi_\rho : G \to \mathbb{C}$ of ρ is defined by

$$\chi_\rho(g) = \mathrm{Tr}\, \rho(g)$$

for all $g \in G$.

Lemma 2.3. *Let χ be the character of a representation ρ of degree d. Then,*

(i) $\chi(1) = d$; *and*
(ii) $\chi(g^{-1}) = \overline{\chi(g)}$ *for all $g \in G$; and*

(iii) $\chi(ghg^{-1}) = \chi(h)$ *for all* $g, h \in G$.

Proof. Items (i) and (iii) follow immediately from the definition of a representation and the properties of the trace operator. For Item (ii), let $g \in G$, and observe that $g \in G$ has finite order, that is, there exists a positive integer m such that $g^m = 1$. Thus, $\rho(g)^m = \rho(1) = I$, where I denotes the identity map on V. But this implies that every eigenvalue λ of $\rho(g)$ must satisfy $\lambda^m = 1$, so the eigenvalues λ are mth roots of unity. Consequently, since the trace of a linear map is the sum of its eigenvalues, we have

$$\overline{\chi(g)} = \overline{\operatorname{Tr}\rho(g)} = \sum \bar{\lambda}_i = \sum \lambda_i^{-1} = \operatorname{Tr}\rho(g)^{-1} = \operatorname{Tr}\rho(g^{-1}) = \chi(g^{-1}).$$

□

Recall that we write $\rho = \rho_1 \oplus \rho_2$ if $\rho : G \to \mathrm{GL}(V)$ is the direct sum of subrepresentations $\rho_i : G \to \mathrm{GL}(V_i)$ acting on subspaces V_i of V such that $V = V_1 \oplus V_2$.

Theorem 2.4. *Let* $\rho_1 : G \to \mathrm{GL}(V_1)$ *and* $\rho_2 : G \to \mathrm{GL}(V_2)$ *be two linear representations of* G*, and let* χ_1 *and* χ_2 *be their characters. Then, the character of* $\rho_1 \oplus \rho_2$ *is* $\chi_1 + \chi_2$.

Proof. Let $B_1 = \{b_i^{(1)}\}_{i=1}^{d_1}$ be a base for V_1 and let $B_2 = \{b_j^{(2)}\}_{i=1}^{d_2}$ be a base for V_2. Thus, the union $B_1 \cup B_2$ is a base for the (external) direct sum $V_1 \oplus V_2$. The trace $\chi(g) = \operatorname{Tr}(\rho_1 \oplus \rho_2)(g)$ relative to $B_1 \cup B_2$ is clearly $\operatorname{Tr}\rho_1(g) + \operatorname{Tr}\rho_2(g) = \chi_1(g) + \chi_2(g)$, which proves the claim since trace is independent of base. □

2.6. Schur's lemma and orthogonality relations.

Theorem 2.5 (Schur's Lemma)**.** *Let* $\rho_i : G \to \mathrm{GL}(V_i)$*,* $i = 1, 2$*, be two irreducible representations of* G *and let* T *be a linear map of* V_1 *into* V_2 *such that* $\rho_2(g)T = T\rho_1(g)$ *for all* $g \in G$*. Then,*

(i) *if* ρ_1 *and* ρ_2 *are inequivalent, we have* $T = 0$.
(ii) *if* $V_1 = V_2$ *and* $\rho_1 = \rho_2$*, then* $T = \lambda I$ *for some* $\lambda \in \mathbb{C}$.

Proof. We prove the contrapositive of (i). Suppose $T \neq 0$. Put $W_1 = \operatorname{Ker} T$. Then, for $w \in W_1$ we have $T\rho_1(g)w = \rho_2(g)Tw = \rho_2(g)0 = 0$ for all $g \in G$. Thus, W_1 is ρ_1-invariant. Since ρ_1 is irreducible, we must have either $W_1 = 0$ or $W_1 = V_1$. The latter alternative is impossible since $T \neq 0$. Next, put $W_2 = \operatorname{Im} T$. Then, for $w = Tv \in W_2$ we have $\rho_2(g)w = T\rho_1(g)v$ for all $g \in G$. Thus, W_2 is ρ_2-invariant and we must have either $W_2 = V_2$ or $W_2 = 0$. The latter is impossible since $T \neq 0$. Now, because $\operatorname{Ker} T = 0$ and $\operatorname{Im} T = V_2$, it must be that T is bijective and hence ρ_1 and ρ_2 are equivalent.

For item (ii) let λ be an eigenvalue of T. Put $T' = T - \lambda I$. Since λ is an eigenvalue, $\operatorname{Ker} T' \neq 0$. Since $\rho_2 T' = T' \rho_1$, item (i) now shows that this is possible only when $T' = 0$, so we must have $T = \lambda I$. □

Corollary 2.6. *Assume* ρ_1 *and* ρ_2 *are as above. Let* T *be a linear map of* V_1 *into* V_2 *and put*

$$T' = \frac{1}{|G|} \sum_{g \in G} \rho_2(g) T \rho_1(g)^{-1}.$$

Then,

(i) *if* ρ_1 *and* ρ_2 *are inequivalent, we have* $T' = 0$.

(ii) *if* $V_1 = V_2$ *and* $\rho_1 = \rho_2$, *then* $T' = \lambda I$ *for* $\lambda = (1/d)\operatorname{Tr} T$, *where* d *is the degree of* ρ_1.

Proof. The following calculation shows that $\rho_2(g)T' = T'\rho_1(g)$ holds for all $g \in G$:

$$\begin{aligned}\rho_2(g)T'\rho_1(g)^{-1} &= \frac{1}{|G|}\sum_{h\in G}\rho_2(g)\rho_2(h)T\rho_1(h)^{-1}\rho_1(g)^{-1}\\ &= \frac{1}{|G|}\sum_{h\in G}\rho_2(gh)T\rho_1(gh)^{-1} = T',\end{aligned}$$

Now, Item (i) follows directly from Item (i) of Schur's Lemma. Similarly, Item (ii) of Schur's Lemma implies that $T' = \lambda I$ for some $\lambda \in \mathbb{C}$. It remains to calculate λ. Observe that

$$\operatorname{Tr} T' = \frac{1}{|G|}\sum_{g\in G}\operatorname{Tr}\rho_1(g)T\rho_1(g)^{-1} = \operatorname{Tr} T$$

and $\operatorname{Tr} T' = \operatorname{Tr}\lambda I = d\lambda$. Solving for λ yields $\lambda = (1/d)\operatorname{Tr} T$. □

Corollary 2.7. *Assume* ρ_1 *and* ρ_2 *are as above. Fix arbitrary bases for* V_1 *and* V_2 *and denote by* $\varrho^{(1)}$ *and* $\varrho^{(2)}$ *the corresponding matrix representations of* ρ_1 *and* ρ_2. *Then,*

(i) *if* ρ_1 *and* ρ_2 *are inequivalent, we have*

(2) $$\sum_{g\in G}\varrho^{(1)}_{ij}(g)\varrho^{(2)}_{rs}(g^{-1}) = 0$$

for all i, j, r, s.

(ii) *if* $V_1 = V_2$ *and* $\rho_1 = \rho_2$, *then*

(3) $$\frac{1}{|G|}\sum_{g\in G}\varrho^{(1)}_{ij}(g)\varrho^{(2)}_{rs}(g^{-1}) = \frac{\delta_{is}\delta_{jr}}{d},$$

where d *is the degree of* $\rho_1 = \rho_2$.

Proof. Assuming the notation of the previous corollary, let (t_{ab}) and (t'_{ab}) be the matrices of T and T' relative to the bases chosen for V_1 and V_2. Straightforward computation gives then

$$t'_{is} = \frac{1}{|G|}\sum_{g,a,b}\varrho^{(2)}_{ia}(g)t_{ab}\varrho^{(1)}_{bs}(g^{-1})$$

for all i, s. Item (i) of the previous corollary gives $t'_{is} = 0$ for all matrices (t_{ab}); in particular this holds for the matrix $t_{ab} = \delta_{aj}\delta_{br}$, which proves Item (i). Similarly, Item (ii) of the previous corollary gives $t'_{is} = \delta_{is}(1/d)\sum_{a,b}t_{ab}\delta_{ab}$ in the case $\rho_1 = \rho_2$ and $V_1 = V_2$. Setting $t_{ab} = \delta_{aj}\delta_{br}$ now gives

$$\frac{1}{|G|}\sum_{g}\varrho^{(2)}_{ij}(g)\varrho^{(1)}_{rs}(g^{-1}) = (1/d)\sum_{a,b}\delta_{is}\delta_{aj}\delta_{br}\delta_{ab} = (1/d)\delta_{is}\delta_{jr}.$$

□

The relations (2) and (3) are known as the *Schur relations.*

Theorem 2.8.

(i) *If* χ *is the character of an irreducible representation, then* $\langle\chi,\chi\rangle = 1$.

(ii) *If χ and χ' are the characters of two inequivalent irreducible representations, then $\langle \chi, \chi' \rangle = 0$.*

Proof. Recall from Lemma 2.3 that $\chi(g^{-1}) = \overline{\chi(g)}$. Thus,

$$\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi'(g^{-1}) = \sum_{i,j} \frac{1}{|G|} \sum_{g \in G} \varrho_{ii}(g)\varrho'_{jj}(g^{-1}) =$$
$$= \begin{cases} 1 & \text{if } \rho \text{ and } \rho' \text{ are equivalent;} \\ 0 & \text{otherwise,} \end{cases}$$

where the last equality follows from the Schur relations (2) and (3). □

Theorem 2.9. *Let ρ be a linear representation of G on V and let $\rho = \rho_1 \oplus \cdots \oplus \rho_n$ be a decomposition of ρ into irreducible subrepresentations. Then, for any irreducible representation ρ' of G on V, the inner product $\langle \chi_\rho, \chi_{\rho'} \rangle$ gives the number of ρ_k equivalent to ρ' in the chosen decomposition.*

Proof. Recall that $\rho = \rho_1 \oplus \cdots \oplus \rho_n$ implies by Theorem 2.4 that $\chi_\rho = \chi_{\rho_1} + \cdots + \chi_{\rho_n}$. Since the ρ_k are irreducible, the claim follows from the previous theorem. □

Corollary 2.10.

(i) *The number of ρ_k equivalent to ρ' does not depend on the chosen decomposition.*

(ii) *Two representations with the same character are equivalent.*

(iii) *Suppose that ρ decomposes as $\rho = \oplus_{k=1}^{n} m_k \rho_k$, where m_k indicates the multiplicity of the irreducible representation ρ_k in ρ, and the ρ_k are pairwise inequivalent. Then,*

$$\langle \chi_\rho, \chi_\rho \rangle = \sum_{k=1}^{n} m_k^2. \tag{4}$$

In particular, $\langle \chi_\rho, \chi_\rho \rangle = 1$ if and only if ρ is irreducible.

2.7. **Number of irreducible representations.** A function $\phi : G \to \mathbb{C}$ is called a *class function* if $\phi(ghg^{-1}) = \phi(h)$ for all $g, h \in G$. Denote by $\mathrm{cf}(G)$ the vector space of all complex-valued class functions on G. Clearly, $\mathrm{cf}(G)$ is a subspace of $\mathbb{C}[G]$.

Theorem 2.11. *The dimension of $\mathrm{cf}(G)$ is equal to the number of conjugacy classes of G.*

Proof. A class function must be constant on each conjugacy class of G; the values on the conjugacy classes may be chosen arbitrarily. □

Corollary 2.12. *The number of inequivalent irreducible representations of G is finite.*

Proof. The character of a representatation of G is a class function by Lemma 2.3. The characters of the inequivalent irreducible representations of G form by Theorem 2.8 an orthonormal set in $\mathrm{cf}(G)$. The cardinality of such a set is bounded by the dimension of $\mathrm{cf}(G)$, which is finite. □

Lemma 2.13. *Let ϕ be a class function on G and let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G. Define a linear map $\hat{\rho}(\phi)$ of V into itself by*

$$\hat{\rho}(\phi) = \sum_{g\in G} \phi(g)\rho(g). \tag{5}$$

Then, if ρ is irreducible and has degree d, we have

$$\hat{\rho}(\phi) = \frac{|G|}{d}\langle \phi, \bar{\chi}_\rho\rangle I,$$

where I denotes the identity map on V.

Proof. Let $g \in G$ and observe that

$$\rho(g)\hat{\rho}(\phi)\rho(g)^{-1} = \sum_{h\in G} \phi(h)\rho(ghg^{-1}) = \sum_{h\in G} \phi(g^{-1}hg)\rho(h) = \hat{\rho}(\phi)$$

since ϕ is a class function. Item (ii) of Corollary 2.6 now implies that

$$\hat{\rho}(\phi) = \frac{1}{|G|}\sum_{g\in G} \rho(g)\hat{\rho}(\phi)\rho(g)^{-1} = \lambda I$$

for $\lambda = (1/d)\,\mathrm{Tr}\,\hat{\rho}(\phi)$. Since

$$\mathrm{Tr}\,\hat{\rho}(\phi) = \sum_{g\in G} \phi(g)\,\mathrm{Tr}\,\rho(g) = \sum_{g\in G} \phi(g)\chi_\rho(g) = |G|\langle \phi, \bar{\chi}_\rho\rangle,$$

the claim follows. □

Theorem 2.14. *The characters of the inequivalent irreducible representations of G form an orthonormal base for $\mathrm{cf}(G)$. [Hence, the number of inequivalent irreducible representations of G is equal to the number of conjugacy classes of G by Theorem 2.11.]*

Proof. By Theorem 2.8 the characters $\chi_1, \dots, \chi_n$ of the inequivalent irreducible representations of G form an orthonormal system in $\mathrm{cf}(G)$, so it suffices to show that their linear span is $\mathrm{cf}(G)$. For this it suffices to show that $\langle \phi, \bar{\chi}_k\rangle = 0$ for all $k = 1, \dots, n$ implies $\phi \equiv 0$. [Note that the χ_k form an orthonormal base if and only if the $\bar{\chi}_i$ do.] Let ϕ be a class function that satisfies $\langle \phi, \bar{\chi}_k\rangle = 0$ for all $k = 1, \dots, n$. Then, any representation ρ of G satisfies $\hat{\rho}(\phi) = 0$ because each of its irreducible constituents ρ_k satisfies $\hat{\rho}_k(\phi) = (|G|/d_k)\langle \phi, \bar{\chi}_k\rangle I = 0$ by Lemma 2.13. In particular, for the left regular representation ρ_{reg} [Recall Example 2.1] we have, for all $g \in G$,

$$\hat{\rho}_{\mathrm{reg}}(g)e_1 = \sum_{g\in G} \phi(g)\rho_{\mathrm{reg}}(g)e_1 = \sum_{g\in G} \phi(g)e_g = 0,$$

which implies $\phi \equiv 0$. □

2.8. **Unitarity.** Recall from linear algebra that a linear map T of a finite dimensional complex inner product space V into itself is *unitary* if it preserves the inner product, that is, for all $v, v' \in V$, $\langle Tv, Tv'\rangle = \langle v, v'\rangle$. This is equivalent to saying that T is bijective and $T^{-1} = T^*$, where T^* is the Hilbert adjoint operator defined by $\langle Tv, v'\rangle = \langle v, T^*v'\rangle$ for all $v, v' \in V$. Recall further that if (t_{ij}) is a matrix of T relative to an orthonormal base of V, then $(\bar{t}_{ji})$ is the matrix of T^* relative to that base. In particular, if T is unitary, then the inverse of (t_{ij}) is simply its complex conjugate transpose $(\bar{t}_{ji})$.

Theorem 2.15. *Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G. Then, there exists an inner product on V relative to which the map $\rho(g)$ is unitary for every $g \in G$.*

Proof. Let $\langle\cdot,\cdot\rangle$ be an inner product on V. [An inner product always exists since V has finite dimension.] It is straightforward to verify that

$$(v|v') = \sum_{g\in G} \langle \rho(g)v, \rho(g)v'\rangle$$

is an inner product on V relative to which any $\rho(g)$ is unitary. □

Let $\rho_k : G \to \mathrm{GL}(V_k)$, $k = 1, 2$, be inequivalent irreducible representations of G and let d_k be the degree of ρ_k.

Corollary 2.16. *For $k = 1, 2$, fix a base for V_k that is orthonormal in an inner product relative to which $\rho_k(g)$ is unitary for every $g \in G$. Denote by $\varrho^{(k)}$ the matrix representation of ρ_k relative to this base. Then,*

$$\langle \varrho^{(k)}_{ij}, \varrho^{(t)}_{rs}\rangle = \frac{\delta_{ir}\delta_{js}\delta_{kt}}{d_k} \tag{6}$$

for all $1 \le k, t \le 2$, $1 \le i, j \le d_k$, and $1 \le r, s \le d_t$.

Proof. By unitarity the entries of the matrices $\varrho^{(k)}(g)$ satisfy

$$\varrho^{(k)}_{ij}(g)^{-1} = \overline{\varrho^{(k)}_{ji}(g)}$$

for all $g \in G$ and $1 \le i, j \le d_k$. Thus, (6) simply combines the Schur relations (2) and (3). □

We call a matrix representation $\varrho^{(k)}$ that meets the conditions of the above corollary a *unitary* matrix representation.

2.9. **Decomposition of the regular representation.** Recall the regular representation ρ_{reg} of G on $\mathbb{C}[G]$ from Example 2.1.

Theorem 2.17. *The character χ_{reg} of the regular representation ρ_{reg} of G satisfies*

$$\chi_{\mathrm{reg}}(g) = \begin{cases} |G| & \text{if } g = 1; \text{ and} \\ 0 & \text{if } g \ne 1. \end{cases} \tag{7}$$

Proof. For every $g \in G$ the matrix of $\rho_{\mathrm{reg}}(g)$ relative to base G is a permutation matrix defined for all $h_1, h_2 \in G$ by

$$\varrho_{h_2h_1}(g) = \begin{cases} 1 & \text{if } h_2 = gh_1; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Now observe that $\chi_{\mathrm{reg}}(g) = \sum_h \varrho_{hh}(g)$ and that $\varrho_{hh}(g) = 1$ if and only if $g = hh^{-1} = 1$. □

Denote by $\chi_1, \ldots, \chi_n$ the characters of the irreducible inequivalent representations $\rho_1, \ldots, \rho_n$ of G, and let $d_1, \ldots, d_n$ denote the degrees of the representations.

Corollary 2.18. *For all $k = 1, \ldots, n$, we have*

$$\langle \chi_{\mathrm{reg}}, \chi_k\rangle = d_k.$$

Proof. Recall from Lemma 2.3 that $\chi_k(1) = d_k$. Now,

$$\langle \chi_{\text{reg}}, \chi_k \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\text{reg}}(g) \overline{\chi_k(g)} = \frac{1}{|G|} \chi_{\text{reg}}(1) \overline{\chi_k(1)} = \frac{1}{|G|} |G| \bar{d}_k = d_k.$$

□

Thus, ρ_{reg} decomposes into the direct sum $\rho_{\text{reg}} = \oplus_{k=1}^{n} d_k \rho_k$.

Corollary 2.19. *The degrees d_k satisfy the relation $\sum_{k=1}^{n} d_k^2 = |G|$.*

Proof. By Theorem 2.9, we have

$$|G| = \chi_{\text{reg}}(1) = \sum_{k=1}^{n} d_k \chi_k(1) = \sum_{k=1}^{n} d_k^2.$$

□

Corollary 2.20. *For $k = 1, \ldots, n$, let $\varrho^{(k)}$ be a unitary matrix representation of ρ_k. Then, the vectors*

$$\bar{\varrho}_{ij}^{(k)} \in \mathbb{C}[G], \qquad 1 \le k \le n, \quad 1 \le i, j \le d_k \tag{8}$$

form an orthogonal base for $\mathbb{C}[G]$.

Proof. Pairwise orthogonality [and hence linear independence] was demonstrated in Corollary 2.16. The vectors span $\mathbb{C}[G]$ by the previous corollary. □

The base (8) incorporates a convenient factorization of $\mathbb{C}[G]$ into ρ_{reg}-invariant subspaces as the following lemma demonstrates. Let $\phi, \psi : G \to \mathbb{C}$. The *convolution* $\phi * \psi$ is defined by

$$(\phi * \psi)(g) = \sum_{h \in G} \phi(gh^{-1}) \psi(h)$$

for each $g \in G$. A direct calculation shows that convolution and the left regular representation ρ_{reg} are connected by

$$\rho_{\text{reg}}(g)\phi(h) = (e_g * \phi)(h) = \phi(g^{-1}h). \tag{9}$$

Lemma 2.21. *For $k = 1, \ldots, n$, let $\varrho^{(k)}$ be a unitary matrix representation that corresponds to ρ_k. Then, for all $g \in G$ and $1 \le i, j \le d_k$,*

$$\rho_{\text{reg}}(g) \bar{\varrho}_{ij}^{(k)} = e_g * \bar{\varrho}_{ij}^{(k)} = \sum_{l=1}^{d_k} \varrho_{li}^{(k)}(g) \bar{\varrho}_{lj}^{(k)}. \tag{10}$$

Proof. Let $g, h \in G$. A direct calculation gives

$$(e_g * \bar{\varrho}_{ij}^{(k)})(h) = \sum_{z \in G} e_g(hz^{-1}) \bar{\varrho}_{ij}^{(k)}(z) = \bar{\varrho}_{ij}^{(k)}(g^{-1}h) = \sum_{l=1}^{d_k} \varrho_{li}^{(k)}(g) \bar{\varrho}_{lj}^{(k)}(h).$$

□

Thus, for all $k = 1, \ldots, n$ and $1 \le j \le d_k$, the vectors $\{\bar{\varrho}_{ij}^{(k)}\}_{1 \le i \le d_k}$ constitute a base for a ρ_{reg}-invariant subspace of $\mathbb{C}[G]$ of dimension d_k.

2.10. **Direct products of groups.** Let V_1 and V_2 be finite-dimensional complex vector spaces. We denote by $V_1 \otimes V_2$ the tensor product of V_1 and V_2.

Let G and G' be finite groups. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G and let $\rho' : G' \to \mathrm{GL}(V')$ be a representation of G'. Denote by $G \times G'$ the direct product of G and G'. Define the representation $\rho \# \rho' : G \times G' \to \mathrm{GL}(V \otimes V')$ by setting

$$(\rho \# \rho')(g, g') = \rho(g) \otimes \rho'(g')$$

for all $(g, g') \in G \times G'$.

Theorem 2.22. *Let $\rho_1, \ldots, \rho_n$ be the inequivalent irreducible representations of G and let $\rho'_1, \ldots, \rho'_{n'}$ be the inequivalent irreducible representations of G'. Then, $\rho_i \# \rho'_j : G \times G' \to \mathrm{GL}(V_i \otimes V_j)$, where $1 \le i \le n$ and $1 \le j \le n'$, is a complete set of inequivalent irreducible representations of $G \times G'$.*

Proof. Since

$$\chi_{\rho_i \# \rho'_j}(g, g') = \mathrm{Tr}\, \rho_i(g) \otimes \rho'_j(g') = \mathrm{Tr}\, \rho_i(g) \cdot \mathrm{Tr}\, \rho'_j(g') = \chi_{\rho_i}(g) \cdot \chi_{\rho'_j}(g'),$$

we have

$$\begin{aligned}
\langle \chi_{\rho_i \# \rho'_j}, \chi_{\rho_k \# \rho'_l} \rangle &= \frac{1}{|G \times G'|} \sum_{g, g'} \chi_{\rho_i \# \rho'_j}(g, g') \overline{\chi_{\rho_k \# \rho'_l}(g, g')} \\
&= \left(\frac{1}{|G|} \sum_{g} \chi_{\rho_i}(g) \overline{\chi_{\rho_k}(g)} \right) \left(\frac{1}{|G'|} \sum_{g'} \chi_{\rho'_j}(g') \overline{\chi_{\rho'_l}(g')} \right) \\
&= \langle \chi_{\rho_i}, \chi_{\rho_k} \rangle \cdot \langle \chi_{\rho'_j}, \chi_{\rho'_l} \rangle = \delta_{ik} \delta_{jl}.
\end{aligned}$$

Thus, the $\chi_{\rho_i \# \rho'_j}$ are irreducible and pairwise inequivalent by Theorem 2.8 and Corollary 2.10. To establish completeness it suffices [recall Corollary 2.19] to show that the squares of the degrees of the representations sum to $|G \times G'|$. The degree of $\rho_i \# \rho'_j$ is clearly $d_i d'_j$, where d_i is the degree of ρ_i and d'_j is the degree of ρ'_j. Thus, $\sum_{i,j} (d_i d'_j)^2 = (\sum_i d_i^2)(\sum_j {d'_j}^2) = |G||G'| = |G \times G'|$. ☐

2.11. **The Fourier transform.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G and let $\phi : G \to \mathbb{C}$. The map

$$\hat{\rho}(\phi) = \sum_{g \in G} \phi(g) \rho(g)$$

defined in Lemma 2.13 is the *Fourier transform* of ϕ relative to ρ.

Theorem 2.23 (Fourier Inversion Formula)**.** *Let $\rho_1, \ldots, \rho_n$ be a complete set of representatives for the irreducible inequivalent representations of G and denote by d_i the degree of ρ_i. Let $\phi : G \to \mathbb{C}$. Then, for all $g \in G$,*

$$\phi(g) = \frac{1}{|G|} \sum_{i=1}^{n} d_i \, \mathrm{Tr}\, \rho_i(g^{-1}) \hat{\rho}_i(\phi). \tag{11}$$

Proof. We have

$$\mathrm{Tr}\, \rho_i(g^{-1}) \hat{\rho}_i(\phi) = \sum_{h \in G} \phi(h) \, \mathrm{Tr}\, \rho_i(g^{-1} h) = \sum_{h \in G} \phi(h) \chi_i(g^{-1} h),$$

so the right hand side of (11) simplifies to

$$\frac{1}{|G|}\sum_{h\in G}\phi(h)\sum_{i=1}^{n} d_i\chi_i(g^{-1}h) = \sum_{h\in G}\phi(h)\frac{1}{|G|}\chi_{\mathrm{reg}}(g^{-1}h) = \phi(g),$$

where the last two equalities follow from $\chi_{\mathrm{reg}} = \sum_i d_i\chi_i$ and (7). □

Theorem 2.24 (Plancherel Formula)**.** *Let $\rho_1, \dots, \rho_n$ be a complete set of representatives for the irreducible inequivalent representations of G and denote by d_i the degree of ρ_i. Let $\phi, \psi : G \to \mathbb{C}$. Then,*

$$\langle \phi, \psi\rangle = \frac{1}{|G|^2}\sum_{i=1}^{n} d_i \operatorname{Tr}\hat{\rho}_i(\phi)\hat{\rho}_i(\psi)^*, \tag{12}$$

where $\hat{\rho}_i(\psi)^*$ denotes the Hilbert adjoint operator, evaluated relative to an inner product in which $\rho_i(g)$ is unitary for all $g \in G$.

Proof. We have

$$\sum_i d_i \operatorname{Tr}\hat{\rho}_i(\phi)\hat{\rho}_i(\psi)^* = \sum_{g,h}\phi(g)\overline{\psi(h)}\sum_i d_i \operatorname{Tr}\rho_i(g)\rho_i(h)^* = |G|\sum_g \phi(g)\overline{\psi(g)},$$

where the second equality follows from $\rho_i(h)^* = \rho_i(h^{-1})$ and $\sum_i d_i \operatorname{Tr}\rho_i(gh^{-1}) = \chi_{\mathrm{reg}}(gh^{-1})$. □

Theorem 2.25. *Let $\phi, \psi : G \to \mathbb{C}$. Then,*

$$\hat{\rho}(\phi * \psi) = \hat{\rho}(\phi)\hat{\rho}(\psi).$$

Proof. A direct calculation gives

$$\begin{aligned}\hat{\rho}(\phi * \psi) &= \sum_{g\in G}(\phi * \psi)(g)\rho(g) = \sum_{g,h\in G}\phi(gh^{-1})\psi(h)\rho(g)\\ &= \sum_{g,h\in G}\phi(g)\psi(h)\rho(gh) = \hat{\rho}(\phi)\hat{\rho}(\psi).\end{aligned}$$

□

3. Representations of certain groups

3.1. **Cyclic and abelian groups.** Denote by $\mathbb{Z}_r$ the cyclic group of order r. We use additive notation for the elements $\{0, 1, 2, \dots, r-1\}$ of $\mathbb{Z}_r$. The conjugacy classes of $\mathbb{Z}_r$ are singleton sets because $x + y - x = y'$ for some $x, y, y' \in \mathbb{Z}_r$ implies $y = y'$. Thus, $\mathbb{Z}_r$ has r irreducible representations, each of degree 1.

Let $w_r = e^{2\pi i/r}$, where i is the imaginary unit. Recall that

$$\sum_{y=0}^{r-1} w_r^{xy} = \begin{cases} r & \text{if } x = 0\text{; and}\\ 0 & \text{if } x \neq 0.\end{cases} \tag{13}$$

Let $V = \mathbb{C}$ be the complex vector space of dimension one. For each $x \in \mathbb{Z}_r$, define $\rho_x : \mathbb{Z}_r \to \mathrm{GL}(V)$ by $\rho_x(y)v = w_r^{xy}v$ for all $y \in \mathbb{Z}_r$ and $v \in V$. The map ρ_x is clearly a linear representation since the inverse of $\rho_x(y)$ is $\rho_x(-y)$ and

$$\rho_x(y + y')v = w_r^{x(y+y')}v = w_r^{xy}w_r^{xy'}v = \rho_x(y)\rho_x(y')v$$

holds for all $y, y' \in \mathbb{Z}_r$ and $v \in V$.

Let χ_x be the character of ρ_x. Clearly, $\chi_x(y) = w_r^{xy}$ for all $y \in \mathbb{Z}_r$. The representations ρ_x are irreducible and pairwise inequivalent by Theorem 2.8 and Corollary 2.10 since the characters satisfy the orthogonality relation

$$\langle \chi_x, \chi_{x'} \rangle = \frac{1}{r}\sum_{y=0}^{r-1} \chi_x(y)\overline{\chi_{x'}(y)} = \frac{1}{r}\sum_{y=0}^{r-1} w_r^{(x-x')y} = \delta_{xx'},$$

where the last equality follows from (13).

Recall from group theory (see e.g. [14]) that every finite Abelian group is isomorphic to a direct product of cyclic groups [of prime power order]. Thus, the results from Section 2.10 enable the construction of the irreducible inequivalent representations of an arbitrary finite Abelian group from the irreducible inequivalent representations of the component cyclic groups.

Let us consider as an example $\mathbb{Z}_r^n$, the n-fold direct product of the cyclic group $\mathbb{Z}_r$. We regard an element $x \in \mathbb{Z}_r^n$ as an n-tuple $x = (x_1, x_2, \ldots, x_n)$ of elements of $\mathbb{Z}_r$.

By Theorem 2.22 the degree-one irreducible representation $\rho_x : \mathbb{Z}_r^n \to \mathrm{GL}(V)$ that corresponds to $x \in \mathbb{Z}_r^n$ satisfies

$$\rho_x(y)v = w_r^{\sum_{i=1}^n x_i y_i} v \tag{14}$$

for all $y \in \mathbb{Z}_r^n$ and $v \in V$. Since the representation has degree one, the character χ_x is "equal" to the representation, that is,

$$\chi_x(y) = w_r^{\sum_{i=1}^n x_i y_i} \tag{15}$$

for all $x, y \in \mathbb{Z}_r^n$.

3.2. **The dihedral group.** Recall that the *dihedral group* D_n, $n \geq 3$, is a group of order $2n$ which is generated by two elements r, s such that

$$r^n = 1, \qquad s^2 = 1, \qquad sr = r^{-1}s.$$

The elements of D_n are easily seen to be $1, r, r^2, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}$.

The conjugacy classes of D_n are different depending on whether n is odd or even. For odd n there are three types of conjugacy classes:

$$\begin{array}{l|c|c}
 & |\cdot| & \# \\
\hline
\{1\} & 1 & 1 \\
\{r^k, r^{n-k}\}, \quad 1 \leq k \leq (n-1)/2 & 2 & (n-1)/2 \\
\{sr^j \ : \ 0 \leq j \leq n-1\} & n & 1
\end{array}.$$

For even n there are five types of conjugacy classes:

$$\begin{array}{l|c|c}
 & |\cdot| & \# \\
\hline
\{1\} & 1 & 1 \\
\{r^k, r^{n-k}\}, \quad 1 \leq k \leq n/2-1 & 2 & n/2-1 \\
\{r^{n/2}\} & 1 & 1 \\
\{sr^{2j+1} \ : \ 0 \leq j \leq n/2-1\} & n/2 & 1 \\
\{sr^{2j} \ : \ 0 \leq j \leq n/2-1\} & n/2 & 1
\end{array}.$$

For odd n the irreducible representations of D_n are:

ρ	d	#
$s^v r^u \mapsto 1$	1	1
$s^v r^u \mapsto (-1)^v$	1	1
$r^u \mapsto \begin{pmatrix} w_n^{ku} & 0 \\ 0 & w_n^{-ku} \end{pmatrix}$, $sr^u \mapsto \begin{pmatrix} 0 & w_n^{-ku} \\ w_n^{ku} & 0 \end{pmatrix}$, $1 \le k \le (n-1)/2$	2	$(n-1)/2$

For even n the irreducible representations of D_n are:

ρ	d	#
$s^v r^u \mapsto 1$	1	1
$s^v r^u \mapsto (-1)^v$	1	1
$s^v r^u \mapsto (-1)^u$	1	1
$s^v r^u \mapsto (-1)^{u+v}$	1	1
$r^u \mapsto \begin{pmatrix} w_n^{ku} & 0 \\ 0 & w_n^{-ku} \end{pmatrix}$, $sr^u \mapsto \begin{pmatrix} 0 & w_n^{-ku} \\ w_n^{ku} & 0 \end{pmatrix}$, $1 \le k \le n/2-1$	2	$n/2-1$

A direct calculation shows that these maps are indeed group homomorphisms as required. Irreducibility and pairwise inequivalence are straightforward to verify with the help of Theorem 2.8, Corollary 2.10, and (13). Since the squares of the degrees d sum to $2n$, the above constitute a complete set of representatives for the irreducible inequivalent representations of D_n.

3.3. **The symmetric group.** The representation theory of the symmetric group is a subfield of its own, with entire books devoted to the subject [9, 4]. Our treatment will be limited to quoting formulas that enable the computation of the degree, character, and unitary representing matrices for the inequivalent irreducible representations of the symmetric group.

We shall briefly recall some basic definitions and terminology for the convenience of the reader. A *permutation* π of a finite set X is a bijection of X onto itself. A point $x \in X$ is *fixed* by π if $\pi(x) = x$; otherwise x is *moved* by π. Two permutations π_1, π_2 are *disjoint* if every point moved by the other is fixed by the other. The *identity permutation* fixes every point $x \in X$. A permutation π is a *k-cycle* if there exist $x_1, \ldots, x_k \in X$ such that π fixes every $x \in X \setminus \{x_1, \ldots, x_k\}$ and

$$\pi(x_1) = x_2, \quad \pi(x_2) = x_3, \quad \ldots, \pi(x_{k-1}) = x_k, \quad \pi(x_k) = x_1.$$

We write $\pi = (x_1\ x_2\ \cdots\ x_k)$ to indicate that π is a k-cycle of the above form. A *transposition* is a 2-cycle. The *product* of two permutations π_1, π_2 is the permutation $\pi_1\pi_2$ defined by $\pi_1\pi_2(x) = \pi_1(\pi_2(x))$ for all $x \in X$. Every permutation π decomposes uniquely [up to ordering of cycles in the product] into a product of pairwise disjoint cycles, where there is exactly one cycle of length 1 for each point fixed by π. The lengths of the cycles in the decomposition form a partition of the integer n, called the *cycle partition* of π.

A *partition* of a positive integer n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_h)$ of positive integers such that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_h$ and $\sum_{i=1}^h \lambda_i = n$. We write $\lambda \vdash n$ to indicate that λ is a partition of n. We denote the total number of partitions of n by $p(n)$.

The *symmetric group* of degree n consists of all permutations of an n-set X with the product of two permutations defined as above. We write S_n for the symmetric group on $X = \{1, \dots, n\}$.

The conjugacy class of a permutation $\pi_0 \in S_n$ is characterized by its cycle partition. Namely, let $\pi \in S_n$. For any k-cycle $(x_1, x_2, \dots, x_k)$ in the cycle decomposition of π_0, we have

$$\pi(x_1, x_2, \dots, x_k)\pi^{-1} = (\pi(x_1), \pi(x_2), \dots, \pi(x_k)).$$

Thus, conjugation by π alters the content of the disjoint cycles in π_0 in an arbitrary way, but the cycle partition remains invariant. Consequently, S_n has $p(n)$ conjugacy classes.

The inequivalent irreducible representations of S_n are conveniently indexed by the partitions of n. We write ρ_λ, χ_λ, and d_λ, respectively, for the irreducible representation, the character, and the degree of the representation associated with $\lambda \vdash n$.

The *Young diagram* of a partition $\lambda = (\lambda_1, \dots, \lambda_h) \vdash n$ is the set

$$[\lambda] = \{(i,j) \ : \ 1 \le i \le h, \ 1 \le j \le \lambda_i\}.$$

We write λ' for the *conjugate* partition of λ defined by $\lambda' = (\lambda'_1, \dots, \lambda'_{h'})$, where

$$h' = \lambda_1, \qquad \lambda'_j = \max\{i \ : \ (i,j) \in [\lambda]\}, \quad 1 \le j \le h'.$$

For example, consider the partition $\lambda = (4, 3, 1, 1)$ of 9. Then,

$$[\lambda] = \begin{matrix} XXXX \\ XXX \\ X \\ X \end{matrix} \qquad [\lambda'] = \begin{matrix} XXXX \\ XX \\ XX \\ X \end{matrix} \qquad \lambda' = (4, 2, 2, 1).$$

Let $(i,j) \in [\lambda]$. The *hook length* h_{ij} is defined by $h_{ij} = \lambda_i - j + \lambda'_j - i + 1$.

Theorem 3.1 (Hook Formula). *Let $\lambda \vdash n$. Then,*

$$d_\lambda = \frac{n!}{\prod_{(i,j)\in[\lambda]} h_{ij}}.$$

For a proof see e.g. [9, Theorem 2.3.21] or [3, pp. 49–50].

Denote by χ_λ^μ the value of the irreducible character χ_λ for a permutation with cycle partition $\mu = (\mu_1, \dots, \mu_t) \vdash n$. The following formula is quoted from [13]:

Theorem 3.2 (Frobenius Character Formula). *Let $(\lambda_1, \dots, \lambda_h), (\mu_1, \dots, \mu_t) \vdash n$ and $l \ge h$. Then, χ_λ^μ equals the coefficient of $\prod_{i=1}^k x_i^{\lambda_i + l - i}$ in*

$$\prod_{1 \le i < j \le l} (x_i - x_j) \prod_{i=1}^{t} (x_1^{\mu_i} + \cdots + x_l^{\mu_i}).$$

See [3, Section 4.10] for a proof. Diaconis and Shahshahani [1] cite [7] for an accessible proof of the following special case for transpositions:

$$\frac{\chi_\lambda^{(2,1,\dots,1)}}{d_\lambda} = \frac{1}{n(n-1)} \sum_{i=1}^{h} (\lambda_j^2 - (2j-1)\lambda_j). \tag{16}$$

For a survey of alternative character formulas, see [13].

A λ-*tableau* is obtained by assigning each $(i,j) \in [\lambda]$ one of the integers $1,\dots,n$, allowing no repeats. For example, two $(4,3,1,1)$-tableaux are

$$\begin{array}{l} 1234 \\ 567 \\ 8 \\ 9 \end{array} , \qquad \begin{array}{l} 4239 \\ 785 \\ 6 \\ 1 \end{array} .$$

A tableau is *standard* if its entries increase along rows and columns. For example, the left $(4,3,1,1)$-tableau above is standard, while the right one is not.

Theorem 3.3. *The number of standard λ-tableaux is d_λ.*

For a proof, see e.g. [9, Corollary 3.1.13].

Let t^λ be a standard λ-tableau. Denote by $t^{\lambda *}$ the standard tableau obtained by deleting the point n from t^λ. The *last letter sequence* $t_1^\lambda,\dots,t_{d_\lambda}^\lambda$ of all the standard λ-tableaux is defined as follows: For every pair t_i^λ, t_j^λ of standard λ-tableux, $i<j$ if and only if either

(i) the point n occurs in t_i^λ in a higher row than in t_j^λ; or
(ii) $t_i^{\lambda *} = t_k^\beta$ and $t_j^{\lambda *} = t_l^\beta$ for some $k<l$ and $\beta \vdash (n-1)$.

For example, the last letter sequence of the 5 standard $(3,2)$-tableaux is

$$\begin{array}{l} 135 \\ 24 \end{array} < \begin{array}{l} 125 \\ 34 \end{array} < \begin{array}{l} 134 \\ 25 \end{array} < \begin{array}{l} 124 \\ 35 \end{array} < \begin{array}{l} 123 \\ 45 \end{array} .$$

Let t_k^λ be a standard λ-tableau, $\lambda \vdash n$. For $r,s \in \{1,2,\dots,n\}$, denote by (i_r,j_r) and (i_s,j_s) the coordinates $(i,j) \in [\lambda]$ of the points r,s in t_k^λ. The *axial distance* $D_k^\lambda(r,s)$ between r and s in t_k^λ is defined by

$$D_k^\lambda(r,s) = (i_s - j_s) - (i_r - j_r).$$

We write $(m-1,m)t_j^\lambda$ for the λ-tableau in which the points $m-1$, m have been interchanged in t_j^λ.

Theorem 3.4. *Let $\lambda \vdash n$ and suppose $t_1^\lambda,\dots,t_{d_\lambda}^\lambda$ is the last letter sequence of the standard λ-tableux. Then, for all transpositions $\tau = (m-1,m)$, $2 \le m \le n$, the linear map $\rho_\lambda(\tau)$ is represented by a unitary matrix $\varrho^{(\lambda)}(\tau)$ whose coefficients are defined by:*

(i) $\varrho_{ii}^{(\lambda)}(\tau) = \pm 1$ *if t_i^λ contains $m-1$ and m in the same row [+1] or column [-1];*
(ii) *if $i<j$ and $t_i^\lambda = (m-1,m)t_j^\lambda$, then $\varrho^{(\lambda)}(\tau)$ contains the following submatrix:*

$$\begin{pmatrix} \varrho_{ii}^{(\lambda)}(\tau) & \varrho_{ij}^{(\lambda)}(\tau) \\ \varrho_{ji}^{(\lambda)}(\tau) & \varrho_{jj}^{(\lambda)}(\tau) \end{pmatrix} = \begin{pmatrix} -D_i^\lambda(m-1,m)^{-1} & \sqrt{1-D_i^\lambda(m-1,m)^{-2}} \\ \sqrt{1-D_i^\lambda(m-1,m)^{-2}} & D_i^\lambda(m-1,m)^{-1} \end{pmatrix};$$

(iii) $\varrho_{ij}^{(\lambda)}(\tau) = 0$ *everywhere else.*

For a proof, see [9, Section 3.4]. The representing matrices of the transpositions $(m-1,m)$ given in the above theorem can be used to construct the representing matrix $\varrho^{(\lambda)}(\pi)$ for any $\pi \in S_n$ since any permutation can be written as a product of transpositions $(m-1,m)$.

4. Cayley graphs

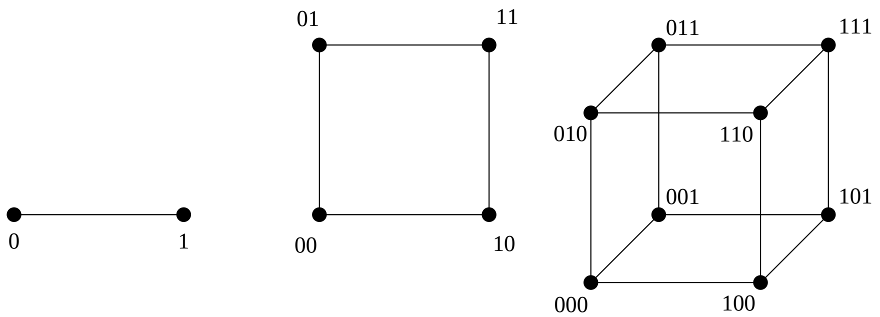
Let G be a finite group and let $\alpha : G \to \mathbb{C}$ be a complex-valued function on G. The *Cayley color graph* $X(G, \alpha)$ is the complete directed graph with vertex set G where each arc $(g_1, g_2) \in G \times G$ is associated a color $\alpha(g_2 g_1^{-1})$.

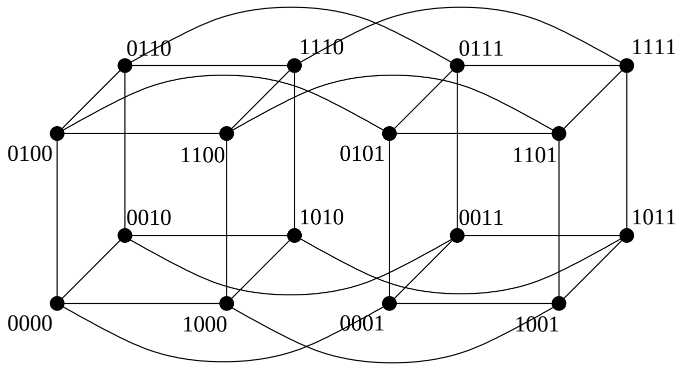
If α is the characteristic function of a subset $S \subseteq G$, that is,

$$\alpha(g) = \begin{cases} 1 & \text{if } g \in S; \text{ and} \\ 0 & \text{if } g \notin S, \end{cases}$$

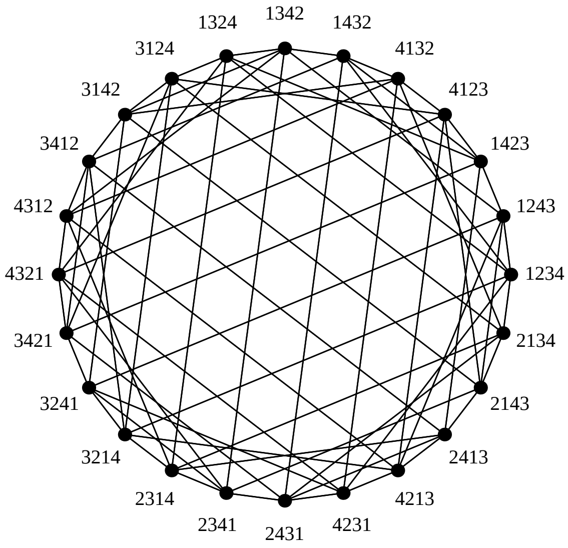
then $X(G, \alpha)$ is a *Cayley digraph* and we denote it by $X(G, S)$. If S is in addition inverse-closed (that is, $S = S^{-1}$) and does not contain the identity of G, then we say that $X(G, S)$ is a *Cayley graph.* In both cases above we regard $X(G, S)$ as the (di)graph formed by deleting the arcs of color 0 from the color graph.

Example 4.1. Let $G = \mathbb{Z}_2^n$, $n \geq 1$, and suppose S consists of all $x \in \mathbb{Z}_2^n$ with exactly one coordinate equal to 1. The resulting Cayley graph $X(\mathbb{Z}_2^n, S)$ is the familiar n-dimensional hypercube. The cases $n = 1, 2, 3, 4$ are depicted below.

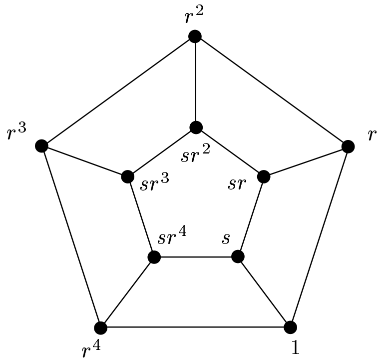




Example 4.2. Let $G = S_4$, the symmetric group on $\{1, 2, 3, 4\}$, and take S to be the set of all transpositions on $\{1, 2, 3, 4\}$. The resulting Cayley graph $X(S_4, S)$ is depicted below.



Example 4.3. Let $G = D_n$, the dihedral group of order $2n$, and take $S = \{r, r^{n-1}, s\}$. The resulting Cayley graph $X(D_n, S)$ is depicted below in the case $n = 5$.



The *adjacency matrix* A of a Cayley color graph $X(G, \alpha)$ is the $|G| \times |G|$ matrix whose row-g_2 column-g_1 entry $a_{g_2 g_1}$ is $\alpha(g_2 g_1^{-1})$ for all $g_1, g_2 \in G$. In case of Cayley digraphs and Cayley graphs the adjacency matrix of $X(G, S)$ corresponds to the standard definition: there is an arc (respectively, edge) from g_1 to g_2 if and only if $a_{g_2 g_1} = \alpha(g_2 g_1^{-1}) = 1$, which is equivalent to saying that there exists an $s \in S$ such that $g_2 = s g_1$.

The following two properties of Cayley graphs are easily verified:

Theorem 4.4. *Let $X(G, S)$ be a Cayley graph. Then,*

(i) *X is regular of degree $|S|$.*
(ii) *X is connected if and only if S generates G.*

Theorem 4.5. *The automorphism group of a Cayley graph is vertex-transitive and contains the **right** regular permutation representation of G as a subgroup.*

Proof. Recall that the right regular permutation representation of G consists of the permutations $\{\pi_g\}_{g \in G}$ of G, where $\pi_g(h) = hg^{-1}$ for all $g, h \in G$. Clearly,

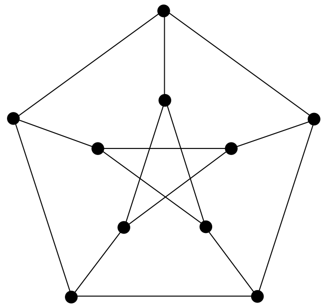
$$\pi_{g_1 g_2}(h) = h g_2^{-1} g_1^{-1} = \pi_{g_2}(h) g_1^{-1} = \pi_{g_1}(\pi_{g_2}(h)),$$

so this collection indeed defines a permutation representation. We shall first prove that $\{\pi_g\}_{g\in G}$ is a subgroup of the automorphism group. For this it suffices to show that for every $h_1, h_2 \in G$, there is an edge between h_1 and h_2 if and only if there is an edge between $\pi_g(h_1)$ and $\pi_g(h_2)$ for every $g \in G$. This is the case since

$$\alpha(h_1 h_2^{-1}) = \alpha(h_1 g^{-1} g h_2^{-1}) = \alpha(\pi_g(h_1)\pi_g(h_2)^{-1})$$

holds for all $g, h_1, h_2 \in G$. Vertex-transitivity is now clear since for every $h_1, h_2 \in G$ there exists a $g \in G$ such that $\pi_g(h_1) = h_2$, namely $g = h_2^{-1} h_1$. □

The Petersen graph below is an example of a graph that has a vertex-transitive automorphism group but is not a Cayley graph.



Theorem 4.6. *A graph X is a Cayley graph if and only if the automorphism group of X has a subgroup that is vertex-transitive and in which no permutation other than the identity fixes a vertex.*

Proof. See e.g. [5, Lemma 3.7.2]. □

5. Eigenvalues and eigenvectors of Cayley color graphs

Let G be a finite group and let $X(G, \alpha)$ be a Cayley color graph. In this section we view the adjacency matrix (a_{gh}) of X as the linear map $A : \mathbb{C}[G] \to \mathbb{C}[G]$ defined by

$$Ae_h = \sum_{g\in G} a_{gh} e_g$$

for all $g, h \in G$.

The following theorem connects the study of Cayley color graphs on G to the representation theory of G.

Theorem 5.1. *Let $X(G, \alpha)$ be a Cayley color graph. Then, the adjacency matrix A [when viewed as a linear map of $\mathbb{C}[G]$ into itself] satisfies*

$$A = \sum_{g\in G} \alpha(g)\rho_{\mathrm{reg}}(g),$$

where ρ_{reg} is the left regular representation of G on $\mathbb{C}[G]$.

Proof. Let $\phi : G \to \mathbb{C}$ and $h \in G$. We have

$$A\phi(h) = \sum_{g\in G} \alpha(hg^{-1})\phi(g) = \sum_{g\in G} \alpha(g)\phi(g^{-1}h) = \sum_{g\in G} \alpha(g)\rho_{\mathrm{reg}}(g)\phi(h).$$

□

Thus, the decomposition of ρ_{reg} into invariant orthogonal subspaces [see Lemma 2.21] yields immediately the following theorem.

Theorem 5.2. *Let $X(G,\alpha)$ be a Cayley color graph, and let $\rho_k : G \to \mathrm{GL}(V_k)$, $k = 1, \dots, n$, be a complete set of irreducible inequivalent representation of G. Let d_k be the degree of ρ_k, and let $\varrho^{(k)}$ be a unitary matrix representation of $\rho^{(k)}$. Then, for all $k = 1, \dots, n$ and $1 \le i, j \le d_k$,*

$$A\bar{\varrho}_{ij}^{(k)} = \sum_{g\in G} \alpha(g)\rho_{\mathrm{reg}}(g)\varrho_{ij}^{(k)} = \sum_{l=1}^{d_k} \left(\sum_{g\in G} \alpha(g)\varrho_{li}^{(k)}(g) \right) \bar{\varrho}_{lj}^{(k)}. \tag{17}$$

Corollary 5.3 (Diaconis and Shahshahani [1])**.** *Let $\mathscr{E}_k$ denote the set of eigenvalues of the linear map $\hat{\rho}_k(\alpha)$. Then,*

(i) *the set of eigenvalues of A equals $\cup_{k=1}^{n} \mathscr{E}_k$; and*
(ii) *if the eigenvalue λ occurs with multiplicity $m_k(\lambda)$ in $\hat{\rho}_k(\alpha)$, then the multiplicity of λ in A is $\sum_{k=1}^{n} d_k m_k(\lambda)$.*

Proof. Equation (17) shows that the vectors $B_j^{(k)} = \{\bar{\rho}_{ij}^{(k)}\}_{1\le i\le d_k}$ span an A-invariant subspace $W_j^{(k)}$ of $\mathbb{C}[G]$ of dimension d_k [cf. Lemma 2.21]. Moreover, (17) shows that A restricted to $W_j^{(k)}$ has [relative to $B_j^{(k)}$] the matrix form

$$\sum_{g\in G} \alpha(g)\varrho^{(k)}(g),$$

which is the matrix of $\hat{\rho}_k(\alpha)$ relative to the basis used to construct the $\varrho^{(k)}(g)$. Now, since $\mathbb{C}[G] = \oplus_{k=1}^{n} \oplus_{j=1}^{d_k} W_j^{(k)}$, the characteristic polynomial $f(\lambda)$ of A and the characteristic polynomials $g_k(\lambda)$ of $\hat{\rho}_k(\alpha)$, $k = 1, \dots, n$, are related by

$$f(\lambda) = \prod_{k=1}^{n} g_k(\lambda)^{d_k}.$$

□

When α is a class function we can say much more, thanks to Lemma 2.13. According to Roichmann [13], the following result on the eigenvalues of A was first proved in Diaconis and Shahshahani [1]. The corresponding eigenvectors were determined [at least] in Rockmore *et al.* [12].

Corollary 5.4. *Let α be a class function. Then, every vector in the orthogonal basis $\{\bar{\varrho}_{ij}^{(k)}\}$ of $\mathbb{C}[G]$ is an eigenvector of A. The eigenvalue associated with $\bar{\varrho}_{ij}^{(k)}$ is*

$$\lambda_k = \frac{|G|}{d_k}\langle \alpha, \bar{\chi}_k\rangle = \frac{1}{d_k}\sum_{g\in G} \alpha(g)\chi_k(g). \tag{18}$$

Proof. Lemma 2.13 gives for a class function α

$$\sum_{g\in G} \alpha(g)\varrho_{li}^{(k)}(g) = \frac{|G|}{d_k}\langle \alpha, \bar{\chi}_k\rangle \delta_{li},$$

which substituted into (17) proves the claim. □

5.1. **Example: Hamming graphs.** The *Hamming graph* $H(n,r)$ is the Cayley graph $X(\mathbb{Z}_r^n, S)$, where S is the set of all elements of $\mathbb{Z}_r^n$ with exactly one nonzero coordinate. In particular, the Hamming graph $H(n,2)$ is the familiar n-dimensional hypercube. [Recall Example 4.1.]

We now determine the eigenvalues and eigenvectors of $H(n,r)$. Since $\mathbb{Z}_r^n$ is abelian, all of its conjugacy classes are singleton sets. This implies that the characteristic function α of S is a class function, and hence we can apply Corollary 5.4.

The eigenvectors of the adjacency matrix A of $H(n,r)$ are thus $\{\bar{\varrho}^{(x)}\}_{x\in\mathbb{Z}_r^n}$, where

$$\bar{\varrho}^{(x)}(y) = w_r^{-\sum_{i=1}^n x_i y_i}, \quad y \in \mathbb{Z}_r^n$$

is the complex conjugate of the 1-dimensional representation (14) associated with $x \in \mathbb{Z}_r^n$. The corresponding eigenvalue λ_x can be evaluated using (18) and the character formula (15): The characteristic function of S is

$$\alpha(y) = \begin{cases} 1 & \text{if there is exactly one } i \text{ such that } y_i \neq 0; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if we denote by $w_H(x)$ the number of nonzero coordinates in x, we have

$$\begin{aligned} \lambda_x = \sum_{y\in\mathbb{Z}_r^n} \alpha(y)\chi_x(y) &= \sum_{i=1}^{n}\sum_{y_i=1}^{p-1} w_r^{x_i y_i} = -n + \sum_{i=1}^{n}\sum_{y_i=0}^{p-1} w_r^{x_i y_i} \\ &= -n + (n - w_H(x))r = (r-1)n - r w_H(x), \end{aligned}$$

where the second last equality follows from (13).

For example, the eigenvectors $\bar{\varrho}^{(x)}$ together with their eigenvalues λ_x for $n=3$, $r=2$ are given in the table below.

y	000	001	010	011	100	101	110	111	λ
$\bar{\varrho}^{(000)}$	1	1	1	1	1	1	1	1	3
$\bar{\varrho}^{(001)}$	1	-1	1	-1	1	-1	1	-1	1
$\bar{\varrho}^{(010)}$	1	1	-1	-1	1	1	-1	-1	1
$\bar{\varrho}^{(011)}$	1	-1	-1	1	1	-1	-1	1	-1
$\bar{\varrho}^{(100)}$	1	1	1	1	-1	-1	-1	-1	1
$\bar{\varrho}^{(101)}$	1	-1	1	-1	-1	1	-1	1	-1
$\bar{\varrho}^{(110)}$	1	1	-1	-1	-1	-1	1	1	-1
$\bar{\varrho}^{(111)}$	1	-1	-1	1	-1	1	1	-1	-3

5.2. **Example: The dihedral group.** We next consider the dihedral group D_n, $n \geq 3$, and the generating set $S = \{r, r^{n-1}, s\}$, which is inverse closed, but not a union of conjugacy classes. [Recall Section 3.2 and Example 4.3.] For simplicity we assume n to be odd. The spectrum of the adjacency matrix of $X(D_n, S)$ can be computed using Corollary 5.3. For the two 1-dimensional representations

$$\varrho^{(1')}(s^v r^u) = 1, \qquad \varrho^{(2')}(s^v r^u) = (-1)^v,$$

we obtain

$$\hat{\varrho}^{(1')}(\alpha) = 3, \qquad \hat{\varrho}^{(2')}(\alpha) = 1.$$

For the remaining $(n-1)/2$ 2-dimensional representations, $1 \leq k \leq (n-1)/2$,

$$\varrho^{(k)}(r^u) = \begin{pmatrix} w_n^{ku} & 0 \\ 0 & w_n^{-ku} \end{pmatrix}, \qquad \varrho^{(k)}(sr^u) = \begin{pmatrix} 0 & w_n^{-ku} \\ w_n^{ku} & 0 \end{pmatrix},$$

we obtain

$$\hat{\varrho}^{(k)}(\alpha) = \begin{pmatrix} w_n^k + w_n^{-k} & 1 \\ 1 & w_n^{-k} + w_n^k \end{pmatrix} = \begin{pmatrix} 2\cos 2\pi k/n & 1 \\ 1 & 2\cos 2\pi k/n \end{pmatrix}.$$

The eigenvalues of the $\hat{\varrho}(\alpha)$ are easily determined to be

λ	$m(\lambda)$	ξ
$\lambda_{1'} = 3$	1	$(1)^T$
$\lambda_{2'} = 1$	1	$(1)^T$
$\lambda_k = 2\cos 2\pi k/n \pm 1, \quad 1 \leq k \leq (n-1)/2$	1, 1	$(1,1)^T, \quad (1,-1)^T$

,

which are the eigenvalues of A by Corollary 5.3. From (17) we obtain that a corresponding set of eigenvectors of A is formed by the linear combinations $\sum_i \xi_i \bar{\varrho}_{ij}^{(k)}$, where $\xi = (\xi_i)^T$ is an eigenvector of $\hat{\varrho}^{(k)}(\alpha)$ corresponding to λ.

5.3. **Example: The symmetric group and transpositions.** Let S be the set of all transpositions of $\{1, 2, \ldots, n\}$. The transpositions constitute a conjugacy class of S_n, so we can use Corollary 5.4 to analyze the spectrum of the adjacency matrix of $X(S_n, S)$. For a partition $\lambda \vdash n$, the dimension d_λ can be determined using the Hook Formula [Theorem 3.1], and the normalized character value $\chi_\lambda^{(2,1,\ldots,1)}/d_\lambda$ can be computed using (16). The corresponding eigenvalue Λ_λ of the adjacency matrix A is then

$$\Lambda_\lambda = \frac{1}{d_\lambda}\binom{n}{2}\chi_\lambda^{(2,1,\ldots,1)}.$$

The table below lists the seven largest Λ_λ.

$\lambda \vdash n$	d_λ	$1 - \chi_\lambda^{(2,1,\ldots,1)}/d_\lambda$	$\binom{n}{2} - \Lambda_\lambda$
(n)	1	0	0
$(n-1,1)$	$n-1$	$2/(n-1)$	n
$(n-2,2)$	$n(n-3)/2$	$4/n$	$2(n-1)$
$(n-2,1,1)$	$(n-1)(n-2)/2$	$4/(n-1)$	$2n$
$(n-3,3)$	$n(n-1)(n-5)/6$	$6(n-2)/(n(n-1))$	$3(n-2)$
$(n-3,2,1)$	$n(n-2)(n-4)/3$	$6/n$	$3(n-1)$
$(n-3,1,1,1)$	$(n-1)(n-2)(n-3)/6$	$6/(n-1)$	$3n$

From the table we can conclude that the spectrum of the graph in Example 4.2 is

$\lambda \vdash 4$	Λ_λ	$m(\Lambda_\lambda) = d_\lambda^2$
(1)	6	1
$(3,1)$	2	9
$(2,2)$	0	4
$(2,1,1)$	-2	9
$(1,1,1,1)$	-6	1

.

A corresponding base of eigenvectors for $\mathbb{C}[S_4]$ could now be computed with the help of Theorem 3.4.

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