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Completeness & Decision Procedures

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“Completeness” and “Decision procedures”

▷ A logic is *complete* if it has a *proof system* that is both sound and complete.

▷ A proof system is a “syntactic” method for establishing *semantic consequence*, e.g.,

“if p and $p \rightarrow q$ are true, then q must be true.”

In other words, q is a semantic consequence of $\{p, (p \rightarrow q)\}$.

▷ A *decision procedure* is an algorithm that determines whether a sentence ϕ is a semantic consequence of a set of sentences Φ.

▷ Not all complete logics are *decidable*, that is, have a decision procedure.

Logic = syntax + semantics

▷ Syntax

$$\mathbf{ML} \stackrel{\text{def}}{=} \mathcal{P} \mid \bot \mid (\mathbf{ML} \rightarrow \mathbf{ML}) \mid \langle R \rangle \mathbf{ML}$$

▷ Semantics via Kripke models and frames:

(Truth in a model)	$\mathcal{M} \models \phi$	$\mathcal{M} = (U, \mathcal{I}, w_0)$;
(Validity in a frame)	$\mathcal{F} \models \phi$	$\mathcal{F} = (U, \mathcal{I})$;
(Universal validity)	$\models \phi$	$\mathcal{F} \models \phi$ for all frames $\mathcal{F}$.

Global semantic consequence

▷ Let $\mathbf{L}$ be a logic (e.g. $\mathbf{ML}$, $\mathbf{CTL}$, $\mathbf{LTL}$) whose semantics are defined via Kripke models.

▷ Let $\Phi \subseteq \mathbf{L}$ be a set of sentences and suppose $\phi \in \mathbf{L}$ is a sentence.

▷ ϕ is a *(global) semantic consequence* of Φ if

$$\mathcal{F} \models \Phi \text{ implies } \mathcal{F} \models \phi \text{ for every frame } \mathcal{F}.$$

We indicate this by writing $\Phi \Vdash \phi$ (or $\Vdash \phi$ if Φ is empty).

▷ $\{p, (p \rightarrow q)\} \Vdash q$

▷ $\{p\} \Vdash [R]p$

▷ $\{(p \rightarrow \mathbf{X}^n q) \ : \ n \in \mathbb{N}\} \Vdash (p \rightarrow \mathbf{G}^* q)$

Proof systems

▷ A *proof system* $\mathscr{P}$ for a logic $\mathbf{L}$ is a **syntactic** method for deciding semantic consequence.

▷ We write $\Phi \vdash \phi$ if we can *prove* ϕ from the premises Φ (using a proof system $\mathscr{P}$).

▷ A proof system need not in general be connected with the semantics of the logic.

▷ A proof system is *sound* if $\Phi \vdash \phi$ implies $\Phi \Vdash \phi$.
"Everything provable from Φ is a semantic consequence of Φ."

▷ A proof system is *complete* if $\Phi \Vdash \phi$ implies $\Phi \vdash \phi$.
"Every semantic consequence of Φ has a proof from Φ."

Deductive proof systems

▷ A *deductive proof system* for a logic $\mathbf{L}$ consists of a set of *axioms* and a set of *deductive rules*.

* An *axiom* is simply a sentence $\phi \in \mathbf{L}$.
* A *deductive rule* is a pair $(\{\phi_1, \dots, \phi_N\}, \psi)$, written

$$\phi_1, \dots, \phi_N \vdash \psi,$$

where $\phi_1, \dots, \phi_N \in \mathbf{L}$ are the *prerequisites* and ψ is the *conclusion*. (The number of prerequisites is always finite.)

Provability in deductive proof systems

▷ Fix any deductive proof system $\mathscr{P}$ for $\mathbf{L}$.

▷ A finite sequence $\phi_1, \dots, \phi_N \in \mathbf{L}$ is a *derivation* of $\phi \in \mathbf{L}$ from the premises $\Phi \subseteq \mathbf{L}$ if $\phi = \phi_N$ and, for every $i = 1, \dots, N$,

1. either ϕ_i is an axiom; or
2. ϕ_i is a premise (i.e. $\phi_i \in \Phi$); or
3. ϕ_i is the conclusion of a deductive rule $\psi_1, \dots, \psi_M \vdash \phi_i$, and $\psi_1, \dots, \psi_M$ appear earlier in the derivation.

▷ We say that ϕ is *provable* from Φ (notation $\Phi \vdash \phi$) if there exists a derivation of ϕ from Φ.

Trivial examples of deductive proof systems

▷ Consider the multimodal logic $\mathbf{ML}$.

1. Take all sentences in $\mathbf{ML}$ as axioms.
2. Assume the axiom set is empty.

▷ Recall that

* a proof system is *sound* if $\Phi \vdash \phi$ implies $\Phi \Vdash \phi$.
* a proof system is *complete* if $\Phi \Vdash \phi$ implies $\Phi \vdash \phi$.

▷ Is either of the "trivial" proof systems above sound?

▷ What about complete?

A sound and complete proof system for $\mathbf{ML}$

(T)	(Propositional tautologies)
(K)	$([R](p \to q) \to ([R]p \to [R]q))$
(MP)	$p,\ (p \to q) \vdash q$
(N)	$p \vdash [R]p$

- ▷ (T) and (K) are axioms.
- ▷ (MP) and (N) are deductive rules.
- ▷ Arbitrary (but systematic) substitution of $\mathbf{ML}$ sentences in place of atomic propositions is allowed to occur.

An example derivation

Let ϕ, ψ be arbitrary $\mathbf{ML}$ sentences and suppose $\Phi = \{(\phi \to \psi)\}$. We derive $([R]\phi \to [R]\psi)$ as follows:

1. $\phi \to \psi$	(GP)
2. $[R](\phi \to \psi)$	(1,N)
3. $([R](\phi \to \psi) \to ([R]\phi \to [R]\psi))$	(K)
4. $([R]\phi \to [R]\psi)$	(2,3,MP)

So, $\{(\phi \to \psi)\} \vdash ([R]\phi \to [R]\psi)$.

Soundness

Theorem. The deductive proof system for $\mathbf{ML}$ is sound.

Proof sketch.

- ▷ Let $\phi_1, \ldots, \phi_M$ be a derivation of ϕ_M from the premises Φ.
- ▷ Let $\mathcal{F} = (U, \mathcal{I})$ be any frame for which $\mathcal{F} \models \Phi$.
- ▷ Proceed by induction: if $\mathcal{F} \models \phi_j$ for all $j = 1, \ldots, i$, conclude that $\mathcal{F} \models \phi_{i+1}$. Then $\mathcal{F} \models \phi_M$ holds eventually.
- ▷ Example: The (N) rule. Suppose that $\phi_{i+1} = [R]\phi_j$, where $j \leq i$. By induction hypothesis $\mathcal{F} \models \phi_j$. Fix any $w \in U$ and consider any w' such that $(w, w') \in \mathcal{I}(R)$. Since $\mathcal{F} \models \phi_j$, we have $(U, \mathcal{I}, w') \models \phi_j$. So, $(U, \mathcal{I}, w) \models [R]\phi_j$ because w' was arbitrary. Because w was arbitrary, $\mathcal{F} \models [R]\phi_j$.

Completeness

Theorem. The deductive proof system for $\mathbf{ML}$ is complete.

Proof sketch.

- ▷ We prove the contrapositive claim $\Phi \nvdash \phi$ implies $\Phi \not\Vdash \phi$.
- ▷ The aim is to construct a *canonical frame* $\mathcal{F}_\Phi = (U, \mathcal{I})$ that satisfies $\mathcal{F}_\Phi \models \Phi$, but for which there exists a $w \in U$ such that $(U, \mathcal{I}, w) \not\models \phi$
- ▷ Then $\mathcal{F}_\Phi$ is the counterexample that demonstrates $\Phi \not\Vdash \phi$.
- ▷ The construction is based on a syntactic notion of consistency with the premises Φ.
- ▷ A set $\Psi \subseteq \mathbf{ML}$ is *consistent* (with Φ) if there exists no finite subset $\{\psi_1, \ldots, \psi_N\} \subseteq \Psi$ such that $\Phi \vdash \neg(\psi_1 \wedge \cdots \wedge \psi_N)$.

- A consistent set Ψ is *maximal* if no proper extension $\Psi' \supset \Psi$ of Ψ is consistent.
- **Lemma (Lindenbaum)**. Every consistent set $\Psi \subseteq \mathbf{ML}$ can be extended to a maximal consistent set.
- **Lemma**. Let $\Psi \subseteq \mathbf{ML}$ be a maximal consistent set. Then $\Phi \subseteq \Psi$ and, for every $\psi \in \mathbf{ML}$, either $\psi \in \Psi$ or $\neg\psi \in \Psi$, but not both.
- Define

$$U \stackrel{\text{def}}{=} \{\Psi \subseteq \mathbf{ML} \ : \ \Psi \text{ is consistent and maximal}\},$$

$$\mathcal{I}(R) \stackrel{\text{def}}{=} \{(\Psi_0, \Psi_1) \in U \times U \ : \ \Psi_0^{[R]} \subseteq \Psi_1\},$$

$$\mathcal{I}(p) \stackrel{\text{def}}{=} \{\Psi \in U \ : \ p \in \Psi\},$$

where $\Psi^{[R]} \stackrel{\text{def}}{=} \{\psi \ : \ [R]\psi \in \Psi\}$.

- **Lemma (Truth)**. For every $\psi \in \mathbf{ML}$ and every $\Psi \in U$, we have $\psi \in \Psi$ if and only if $\Psi \models \psi$.
- So, since $\Phi \subseteq \Psi$ for every $\Psi \in U$, we have $\mathcal{F}_\Phi \models \Phi$.
- Recall that we assume $\Phi \nvdash \phi$.
- So, $\{\neg\phi\}$ must be consistent. (Otherwise $\Phi \vdash \neg(\neg\phi)$, i.e., $\Phi \vdash \phi$.)
- Let $\Psi_0 \in U$ be any maximal consistent extension of $\{\neg\phi\}$.
- Then, since $\neg\phi \in \Psi_0$, we have $\phi \notin \Psi_0$.
- Consequently $(U, \mathcal{I}, \Psi_0) \not\models \phi$ and hence $\mathcal{F}_\Phi \not\models \phi$.

Consequences of the completeness proof

- For any premise set $\Phi \subseteq \mathbf{ML}$ there exists a canonical frame $\mathcal{F}_\Phi = (U, \mathcal{I})$ that has the following property:

$$\Phi \not\Vdash \phi \text{ if and only if } \exists\ \Psi \in U \text{ such that } (U, \mathcal{I}, \Psi) \not\models \phi.$$

"$\Rightarrow$" $\Phi \not\Vdash \phi$ implies $\Phi \nvdash \phi$ by the soundness theorem, so Ψ exists.

"$\Leftarrow$" Clear since $\mathcal{F}_\Phi \models \Phi$ by construction.

- So, it suffices to consider only the canonical frame $\mathcal{F}_\Phi$ to determine whether $\Phi \Vdash \phi$.
- Unfortunately, the canonical frame is uncountably infinite, and the problem of determining whether $\Phi \Vdash \phi$ for arbitrary Φ, ϕ is undecidable.
- For finite Φ the problem becomes decidable. (More on this later.)

Monomodal logic with transitive closure

- Syntax

$$\mathbf{ML}_1^+ \stackrel{\text{def}}{=} \mathcal{P} \mid \bot \mid (\mathbf{ML}_1^+ \to \mathbf{ML}_1^+) \mid \mathbf{X}\,\mathbf{ML}_1^+ \mid \mathbf{F}^*\mathbf{ML}_1^+$$

- Semantics via (restricted) Kripke models and frames.
- $\mathbf{X}\phi \stackrel{\text{def}}{=} \langle \prec \rangle \phi$, $\mathbf{F}^*\phi \stackrel{\text{def}}{=} (\phi \vee \langle < \rangle \phi)$.
- Restriction on models and frames:

$$\mathcal{I}(<) = \mathcal{I}(\prec)^+,$$

where $\mathcal{I}(\prec)^+$ denotes the transitive closure of $\mathcal{I}(\prec)$.

- Syntactic abbreviations:

$$\mathbf{X}\phi \stackrel{\text{def}}{=} \neg\mathbf{X}\neg\phi, \qquad \mathbf{G}^*\phi \stackrel{\text{def}}{=} \neg\mathbf{F}^*\neg\phi.$$

A sound and (weakly) complete proof system for $\mathbf{ML}_1^+$

(T)	(Propositional tautologies)
(K)	$(\mathbf{X}(p \to q) \to (\mathbf{X}p \to \mathbf{X}q))$
(Rec)	$\mathbf{G}^*p \to (p \land \mathbf{X}\mathbf{G}^*p)$
(MP)	$p,\ (p \to q) \vdash q$
(N)	$p \vdash \mathbf{X}p$
(Ind)	$(p \to (q \land \mathbf{X}p)) \vdash (p \to \mathbf{G}^*q)$

- ▷ (T), (K) and (Rec) are axioms.
- ▷ (MP), (N) and (Ind) are deductive rules.
- ▷ Arbitrary (but systematic) substitution of $\mathbf{ML}_1^+$ sentences in place of atomic propositions is allowed to occur.

An example derivation

Let ϕ be an arbitrary $\mathbf{ML}_1^+$ sentence and suppose that $\Phi = \{\phi\}$. We derive $G^*\phi$ as follows:

1.	ϕ	(GP)
2.	$\mathbf{X}\phi$	(1,N)
3.	$(\phi \to (\mathbf{X}\phi \to (\phi \to (\phi \land \mathbf{X}\phi))))$	(T)
4.	$(\mathbf{X}\phi \to (\phi \to (\phi \land \mathbf{X}\phi)))$	(1,3,MP)
5.	$(\phi \to (\phi \land \mathbf{X}\phi))$	(2,4,MP)
6.	$\phi \to \mathbf{G}^*\phi$	(5,Ind)
7.	$\mathbf{G}^*\phi$	(1,6,MP)

$\mathbf{ML}_1^+$ is noncompact

- ▷ Let $\Phi = \{(p \to \mathbf{X}^n q)\ :\ n \in \mathbb{N}\}$ and $\phi = (p \to \mathbf{G}^*q)$.
- ▷ Now $\Phi \Vdash \phi$, but for every finite $\Phi' \subset \Phi$ it holds that $\Phi' \nVdash \phi$.
- ▷ Consider any deductive proof system for $\mathbf{ML}_1^+$ that is sound.
- ▷ Derivations are *finite* sequences that in particular use a finite number of premises. Consequently, if $\Phi \vdash \phi$, then there exists a finite $\Phi' \subset \Phi$ such that $\Phi' \vdash \phi$.
- ▷ Since the proof system is sound, $\Phi' \Vdash \phi$, a contradiction.
- ▷ So, $\Phi \nvdash \phi$.

Completeness

- ▷ $\mathbf{ML}_1^+$ admits no deductive proof system that is both sound and complete.
- ▷ The notion of completeness has to be relaxed.

Definition. A deductive proof system is *weakly complete* if $\Phi \Vdash \phi$ implies $\Phi \vdash \phi$ whenever Φ is **finite**.

Theorem (Deduction). Let $\psi, \phi \in \mathbf{ML}_1^+$. Then, $\psi \Vdash \phi$ if and only if $\Vdash \mathbf{G}^*\psi \to \phi$.

Proposition. A deductive proof system for $\mathbf{ML}_1^+$ is weakly complete if and only if $\Vdash \phi$ implies $\vdash \phi$.

Completeness

Theorem. The $\mathbf{ML}_1^+$ proof system is weakly complete.

Proof sketch.

- ▷ We again prove the contrapositive claim $\nvdash \phi$ implies $\nVdash \phi$.
- ▷ It suffices to construct a model $\mathcal{M} = (U, \mathcal{I}, w)$ such that $\mathcal{M} \not\models \phi$.
- ▷ The model is again based on syntactic consistency.
- ▷ A set $\Psi \subseteq \mathbf{ML}_1^+$ is *consistent* if there exists no finite subset $\{\psi_1, \ldots, \psi_n\} \subseteq \Psi$ such that $\vdash \neg(\psi_1 \wedge \cdots \wedge \psi_n)$.

- ▷ Let $\phi \in \mathbf{ML}_1^+$. The set of *extended subformulas* of ϕ, denoted $\mathrm{ESF}(\phi)$, is the minimal set of formulas that satisfies
  1. $\phi \in \mathrm{ESF}(\phi)$.
  2. If $(\psi_1 \to \psi_2) \in \mathrm{ESF}(\phi)$, then $\psi_1 \in \mathrm{ESF}(\phi)$ and $\psi_2 \in \mathrm{ESF}(\phi)$.
  3. If $\mathbf{X}\psi \in \mathrm{ESF}(\phi)$, then $\psi \in \mathrm{ESF}(\phi)$.
  4. If $\mathbf{F}^*\psi \in \mathrm{ESF}(\phi)$, then $\psi \in \mathrm{ESF}(\phi)$ and $\mathbf{XF}^*\psi \in \mathrm{ESF}(\phi)$.
- ▷ The set $\mathrm{ESF}(\phi)$ is finite for every $\phi \in \mathbf{ML}_1^+$.
- ▷ A consistent set $\Psi \subseteq \mathrm{ESF}(\phi) \cup \neg\mathrm{ESF}(\phi)$ is *maximal* if either $\psi \in \Psi$ or $\neg\psi \in \Psi$ for every $\psi \in \mathrm{ESF}(\phi)$.
- ▷ **Lemma**. Every consistent set $\Psi \subseteq \mathrm{ESF}(\phi) \cup \neg\mathrm{ESF}(\phi)$ can be extended to a maximal consistent set.

- ▷ Define

$$U \stackrel{\text{def}}{=} \{\Psi \subseteq \mathrm{ESF}(\phi) \cup \neg\mathrm{ESF}(\phi) \;:\; \Psi \text{ is consistent and maximal}\},$$

$$\mathcal{I}(\prec) \stackrel{\text{def}}{=} \{(\Psi_0, \Psi_1) \in U \times U \;:\; \Psi_0^{\neg\mathbf{X}} \subseteq \Psi_1\},$$

$$\mathcal{I}(p) \stackrel{\text{def}}{=} \{\Psi \in U \;:\; p \in \Psi\},$$

where $\Psi^{\neg\mathbf{X}} \stackrel{\text{def}}{=} \{\neg\psi \;:\; \neg\mathbf{X}\psi \in \Psi\}$.

- ▷ **Lemma (Truth)**. For every $\psi \in \mathrm{ESF}(\phi)$ and every $\Psi \in U$, we have $\psi \in \Psi$ if and only if $\Psi \models \psi$.
- ▷ Recall that we assume $\nvdash \phi$.
- ▷ So, $\{\neg\phi\} \subseteq \neg\mathrm{ESF}(\phi)$ must be consistent.
- ▷ Let $\Psi_0 \in U$ be any maximal consistent extension of $\{\neg\phi\}$.
- ▷ Then, since $\neg\phi \in \Psi_0$, we have $(U, \mathcal{I}, \Psi_0) \not\models \phi$.

Consequences of the completeness proof

- ▷ Let $\phi \in \mathbf{ML}_1^+$. Then,

  $\nVdash \phi$ if and only if $\exists$ **finite** $(U, \mathcal{I}, w)$ such that $(U, \mathcal{I}, w) \not\models \phi$.

"$\Rightarrow$" $\nVdash \phi$ implies $\nvdash \phi$ by the soundness theorem, so $(U, \mathcal{I}, w)$ exists.

"$\Leftarrow$" Trivial since $(U, \mathcal{I}, w) \not\models \phi$ implies $\nVdash \phi$.

- ▷ The size of the finite model is bounded by

$$|U| \leq 2^{|\mathrm{ESF}(\phi)|}.$$

- ▷ The problem of determining whether $\Vdash \phi$ is decidable: Exhaustively search through all models of size $\leq 2^{|\mathrm{ESF}(\phi)|}$.

Other logics

▷ Sound and (weakly) complete deductive proof systems exist for

- **CTL**, **LTL** with/without past operators.
- **LTL** with natural models.
- **qTL**, μ**TL**.

Decision procedures

▷ A *decision procedure* for a logic $\mathbf{L}$ is an algorithm that determines for a finite $\Phi \subseteq \mathbf{L}$ and a $\phi \in \mathbf{L}$ whether $\Phi \Vdash \phi$.

▷ In practice, the algorithms determine *satisfiability*.

- ϕ is *satisfiable* subject to premises Φ if there exists a frame $\mathcal{F} = (U, \mathcal{I})$ and a state $w \in U$ such that $\mathcal{F} \models \Phi$ and $(U, \mathcal{I}, w) \models \phi$.

▷ $\Phi \Vdash \phi$ if and only if $\neg\phi$ is unsatisfiable subject to Φ.

ML decision procedure (1/3)

▷ Let $\Phi \subseteq \mathbf{ML}$ be finite and let $\phi \in \mathbf{ML}$.

▷ Denote by SF the set of all subformulas of the formulas in $\Phi \cup \{\phi\}$.

▷ A subset $w \subseteq \mathrm{SF}$ is *propositionally consistent* if

1. $\perp \notin w$; and
2. if $(\psi_1 \to \psi_2) \in \mathrm{SF}$, then
   $(\psi_1 \to \psi_2) \in w$ if and only if $\psi_1 \notin w$ or $\psi_2 \in w$.
3. if $\neg\psi \in \mathrm{SF}$, then $\neg\psi \in w$ if and only if $\psi \notin w$.
4. if $(\psi_1 \vee \psi_2) \in \mathrm{SF}$, then
   $(\psi_1 \vee \psi_2) \in w$ if and only if $\psi_1 \in w$ or $\psi_2 \in w$.
5. if $(\psi_1 \wedge \psi_2) \in \mathrm{SF}$, then
   $(\psi_1 \wedge \psi_2) \in w$ if and only if $\psi_1 \in w$ and $\psi_2 \in w$.

ML decision procedure (2/3)

▷ Take as U the set of all $w \subseteq \mathrm{SF}$ that satisfy

1. $w \supseteq \Phi$; and
2. w is propositionally consistent.

▷ Take $\mathcal{I}(R) = U \times U$. (Only $R \in \mathcal{R}$ that appear in SF need to be considered.)

▷ Now remove repeatedly *bad points* and *bad arcs* until none exist.

▷ If U contains a state w with $\phi \in w$, then output “satisfiable”; otherwise output “unsatisfiable.”

$\mathbf{ML}$ decision procedure (3/3)

▷ Bad arcs and points are defined as follows:

* An *arc* $(w, w') \in \mathcal{I}(R)$ *is bad* if $\langle R \rangle \psi \notin w$ but $\psi \in w'$.
* A *point* $w \in U$ *is bad* if $\langle R \rangle \psi \in w$ but $\psi \notin w'$ for all $w' \in U$ such that $(w, w') \in \mathcal{I}(R)$.

▷ The above procedure is a sound and weakly complete proof system for $\mathbf{ML}$.

* For finite Φ, let $\Phi \vdash \phi$ if and only if the procedure outputs "unsatisfiable" on input $\Phi, \neg\phi$.
* For finite Φ, $\Phi \Vdash \phi$ if and only if $\Phi \vdash \phi$.

$\mathbf{ML}_1^+$ decision procedure (1/2)

▷ Let $\Phi \subseteq \mathbf{ML}_1^+$ be finite and let $\phi \in \mathbf{ML}_1^+$.

▷ Let ESF denote the set of *extended* subformulas of $\Phi \cup \{\phi\}$.

* Both $\mathbf{XF}^*\psi$ and ψ are extended subformulas of $\mathbf{F}^*\psi$.

▷ Take as U the set of all $w \subseteq \mathrm{ESF}$ that satisfy

1. $w \supseteq \Phi$; and
2. w is propositinally consistent.

▷ Take $\mathcal{I}(\prec) = U \times U$.

▷ Now remove repeatedly *bad points* and *bad arcs* until none exist.

▷ If U contains a state w with $\phi \in w$, then output "satisfiable"; otherwise output "unsatisfiable."

$\mathbf{ML}_1^+$ decision procedure (2/2)

▷ Recall $\mathbf{ML}_1^+$ axiom

$$(\mathbf{G}^*q \to (q \wedge \mathbf{X}\mathbf{G}^*q)) \qquad \rightsquigarrow \qquad ((q \vee \mathbf{XF}^*q) \to \mathbf{F}^*q)$$

▷ A *point* $w \in U$ *is bad* if

1. $\mathbf{X}\psi \in w$ but $\psi \notin w'$ for all $w' \in U$ such that $(w, w') \in \mathcal{I}(\prec)$; or
2. $\mathbf{F}^*\psi \notin w$ but $\psi \in w$; or
3. $\mathbf{F}^*\psi \in w$ but $\psi \notin w$ and no point reachable from w contains ψ.

▷ An *arc* $(w, w') \in \mathcal{I}(\prec)$ *is bad* if

1. $\mathbf{X}\psi \notin w$ but $\psi \in w'$; or
2. $\mathbf{F}^*\psi \notin w$ but $\mathbf{F}^*\psi \in w'$.

Efficiency and implementation

▷ The number of propositionally consistent sets that contain Φ is in general exponential (in $|\mathrm{SF}|$).

▷ The $\mathbf{ML}$ and $\mathbf{ML}_1^+$ decision algorithms require worst case exponential time.

▷ A large number of propositionally consistent sets need to be stored.

▷ Either bottom-up or top-down construction possible

* top-down: remove states and arcs until a satisfying model is reached.
* bottom-up: add states and arcs until a satisfying model is reached. (Problem: what to add $\Rightarrow$ backtracking)

▷ Bottom-up more suitable for linear time (and natural models).

Satisfiability algorithms for natural models

▷ Natural model: $((w_0, w_1, \dots), \mathcal{I}, w_0)$, where $w_i \prec w_{i+1}$ for all i.

▷ Deterministic monomodal logic.

- ∗ One modal operator (e.g. $\mathbf{X}$).
- ∗ Each state has at most one successor.

▷ Satisfiability of ϕ subject to Φ and linear models:

- ∗ Construct $w_0, w_1, \dots$ step by step using backtracking search.
- ∗ Initial state w_0:
  Consider all prop. consistent $w_0 \subseteq \mathrm{SF}$ with $\Phi \subseteq w_0$ and $\phi \in w_0$.
- ∗ Search step $i \rightsquigarrow i + 1$:
  Given $w_i \subseteq \mathrm{SF}$ as input, attempt to construct a successor $w_{i+1} \subseteq \mathrm{SF}$ so that all *future obligations* are fulfilled.

Obligations in constructing w_{i+1}

1. *Positive* future obligations: $\psi \in w_{i+1}$ for all sentences $\mathbf{X}\psi \in w_i$.
2. *Negative* future obligations: $\psi \notin w_{i+1}$ for all sentences $\neg\mathbf{X}\psi \in w_i$.
3. *Premises*: $\Phi \subseteq w_{i+1}$.
4. *Consistency*: w_{i+1} must be propositionally consistent.

Termination

▷ No positive obligations ⇒ the sequence $(w_0, \dots, w_i)$ is a model.

▷ w_{i+1} is identical to a w_j constructed earlier ⇒ the sequence $(w_0, \dots, w_{j-1}) \circ (w_j, \dots, w_i)^\omega$ is a model.

▷ Finite number of $w \subseteq \mathrm{SF}$ ⇒ algorithm always terminates.