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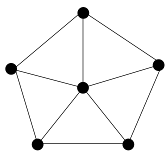
T-79.250 Combinatorial Models and Stochastic Algorithms (4 cr)
Exam Tue 25 Oct 2005, 1–4 p.m.

Permitted material at exam: lecture notes, any personal handwritten notes, tutorial problems and their solutions; calculator.

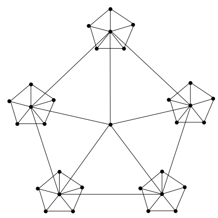
Write down on each answer sheet:
- Your name, department, and study book number
- The text: "T-79.250 Combinatorial Models and Stochastic Algorithms 25.10.2005"
- The total number of answer sheets you are submitting for grading

1. A *k-wheel*, denoted W_k, is a graph that consists of a k-cycle of nodes, each connected by an edge ("spoke") to a central node (the "hub").[1] Thus, the following is a 5-wheel W_5:

   Prove that the graph property "G contains a k-wheel" has a threshold function for any fixed $k \geq 3$, and compute it. *7p.*

2. A *k-wheel of order r*, denoted W_k^r, is defined as follows: a W_k^1 is an ordinary k-wheel W_k, and for $r > 1$ a W_k^r consists of k copies of a W_k^{r-1} whose hubs, together with a new centre node, form a k-wheel. Thus, the following is a 5-wheel of order 2:

   Determine the number of nodes and edges in a k-wheel of order r, and compute the clustering coefficient $\mathcal{C}(W_k^r)$. *7p.*

3. Recall the following results from problem 5 of tutorial 3:

   Consider a random walk on an undirected graph $G = (V, E)$, where transitions are made from each node u to an adjacent node with uniform probability β/d, where d is the maximum degree of any node in G and $\beta \leq 1$ is a positive constant. In addition, each node u has a self-loop probability of $1 - \beta \deg(u)/d$. If G is connected and $\beta < 1$, then the corresponding Markov chain

[1] Actually, in standard graph-theory terminology our W_k's would be called $(k+1)$-wheels.

$\mathcal{M}_G$ is regular and reversible, with uniform stationary distribution. Moreover, the conductance of $\mathcal{M}_G$ is given by the formula

$$\Phi = \beta\mu(G)/d,$$

where $\mu(G)$, the *edge magnification* of G, is defined as

$$\mu(G) = \min_{0<|U|\leq|V|/2} \frac{|\partial(U)|}{|U|},$$

where $\partial(U) = \{\{u,v\} \in E \mid u \in U, v \notin U\}$.

Based on these results, design a Markov chain for sampling (approximately) uniformly at random a node from a k-wheel of order r. Estimate the number of steps required for the sampling process to converge to within a relative pointwise distance of ε of the uniform distribution. (Make some reasonable argument for the value of the relevant edge magnification. You do not need to provide a detailed proof.) *8p.*

4. Consider the NP-complete *Spin Glass Ground State* decision problem: given a symmetric interaction matrix $J \in \{+1,-1\}^{N\times N}$, is there a state $\sigma \in \{+1,-1\}^N$ of the N spins in the system such that *all* the $M = \binom{N}{2}$ bonds are satisfied, i.e. such that the Hamiltonian achieves its minimum value

$$H(\sigma) = -\sum_{i\sim j} J_{ij} S_i S_j = -M?$$

Assuming that the antiferromagnetic bonds (negative interactions) $J_{ij} = -1$ constitute some fraction α of all the bonds, make an educated guess concerning the value of α for which the problem is most difficult to solve, in the ensemble of systems with N spins and $M = \binom{N}{2}$ bonds, out of which αM are chosen to be antiferromagnetic uniformly at random. (*Hint:* It may be easier to consider the a.f.m. bond density α initially as a function of N rather than M.) *8p.*

Total 30p.