Helsinki University of Technology
Laboratory for Theoretical Computer Science
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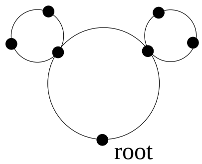
T-79.250 Combinatorial Models and Stochastic Algorithms (4 cr)
Exam Wed 4 May 2005, 12–3 p.m.

Permitted material at exam: lecture notes, any personal handwritten notes, tutorial problems and their solutions; calculator.

Write down on each answer sheet:
- Your name, department, and study book number
- The text: "T-79.250 Combinatorial Models and Stochastic Algorithms 4.5.2005"
- The total number of answer sheets you are submitting for grading

1. An *MM-graph of order k*, denoted MM_k, consists of a base triangle (3-cycle), one of whose nodes is the graph's *root*, and the other two nodes are identified with the roots of two MM-graphs of order $k-1$. Thus, MM_0 is just a single node, MM_1 is a simple triangle with a distinguished root node, and MM_2 is the following graph:

   Determine the number of nodes and edges in an MM-graph of order k, and compute the clustering coefficient $\mathcal{C}(\mathrm{MM}_k)$. *7p.*

2. Recall the following results from problem 5 of tutorial 3:

   Consider a random walk on an undirected graph $G = (V, E)$, where transitions are made from each node u to an adjacent node with uniform probability β/d, where d is the maximum degree of any node in G and $\beta \leq 1$ is a positive constant. In addition, each node u has a self-loop probability of $1 - \beta \deg(u)/d$. If G is connected and $\beta < 1$, then the corresponding Markov chain $\mathcal{M}_G$ is regular and reversible, with uniform stationary distribution. Moreover, the conductance of $\mathcal{M}_G$ is given by the formula

   $$\Phi = \beta\mu(G)/d,$$

   where $\mu(G)$, the *edge magnification* of G, is defined as

   $$\mu(G) = \min_{0<|U|\leq|V|/2} \frac{|\partial(U)|}{|U|},$$

   where $\partial(U) = \{\{u, v\} \in E \mid u \in U, v \notin U\}$.

   Based on these results, design a Markov chain for sampling (approximately) uniformly at random a node from an MM-graph of order k. Estimate the number of steps required for the sampling process to converge to within a relative pointwise distance of ε of the uniform distribution. *8p.*

3. Consider the NP-complete *Graph k-Colouring* problem: Given a graph $G = (V, E)$, is there a node colouring map $\chi : V \to \{1, \ldots, k\}$ such that no two neighbouring nodes get the same colour, i.e. such that $\{i, j\} \in E \Rightarrow \chi(i) \neq \chi(j)$? Make an educated guess concerning the value of $p = p_k(n)$ for which Graph k-Colouring is most difficult to solve in the family of $\mathcal{G}(n, p)$ random graphs. *7p.*

4. Formulate the task of finding good graph k-colourings as a minimisation task, and design a simulated annealing approach to solving it. Present a cooling schedule that would guarantee that the algorithm eventually finds (with probability 1) a minimum-conflict colouring in the case of graphs with maximum node degree Δ. *8p.*

Total 30p.