T-79.250 **Spring 2005**

Combinatorial Models and Stochastic Algorithms

Tutorial 8, March 18

Problems

1. Let Q be a monotone graph property and denote by $P_n^Q(p)$ the probability that a random graph $G \in \mathcal{G}(n,p)$ has property Q. Prove that $P_n^Q(p)$ is a continuous, increasing function of p. (*Hint:* The result may not be as obvious as it seems. One approach is to write down an explicit expression for the probability that a subgraph with exactly k edges has property Q, sum over all k, and differentiate with respect to p. Other, more purely combinatorial approaches are also possible.)

2. Let X be the number of cycles in a random graph $G \in \mathcal{G}(n,p)$, where $p = c/n$. Find an exact formula for $E[X]$, and estimate the asymptotics of $E[X]$ when (a) $c < 1$ and (b) $c = 1$.

3. Consider the space Ω_n of random equiprobable permutations of $[n] = \{1, \ldots, n\}$. A permutation $\pi \in \Omega_n$ contains an *increasing subsequence of length* k, if there are indices $i_1 < \cdots < i_k$ such that $\pi(i_1) < \cdots < \pi(i_k)$.

   (a) Show that almost no permutation $\pi \in \Omega_n$ contains an increasing subsequence of length at least $e\sqrt{n}$. (*Hint:* Let $I_k(\pi)$ be the number of increasing subsequences of length k contained in π. Estimate $E[I_k]$.)

   (b) Denote the length of a maximal increasing subsequence contained in a permutation π by $I(\pi)$, and correspondingly the length of a maximal *decreasing* subsequence by $D(\pi)$. Erdős and Szekeres proved in 1935 that $I(\pi)D(\pi) \geq n$ for any permutation π of $[n]$, (The proof is simple & elegant; think about it or look it up in any combinatorics textbook under "Ramsey theory.") Deduce from this result and the result of part (a) that almost every permutation $\pi \in \Omega_n$ contains an increasing subsequence of length at least $\sqrt{n}/e$.