T-79.250 **Spring 2005**
Combinatorial Models and Stochastic Algorithms
Tutorial 6, March 4
Problems

1. Show that in a canonical ensemble of microstates (i.e. w.r.t. the Gibbs probability distribution), one obtains

$$\langle H^2 \rangle = \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta^2}$$

and, using this result, that the *specific heat* of the system,

$$C_V = \frac{\partial \langle H \rangle}{\partial T},$$

satisfies

$$C_V = \frac{\beta}{T}[\langle H^2 \rangle - \langle H \rangle^2],$$

i.e. indicates the fluctuations around the average energy value.

2. Compute the thermodynamic potential (average energy), specific heat, and entropy of a decoupled ($J = 0$) Ising system with N spins at temperature T and external field h.

3. Same as problems 1 and 2, but with respect to *magnetic susceptibility*

$$\chi = \frac{\partial \langle M \rangle}{\partial h} = \beta[\langle M^2 \rangle - \langle M \rangle^2]$$

in place of specific heat.

4. Construct a 5-bit Hopfield associative memory network for the patterns $(+,+,+,+,+)$, $(+,-,-,+,-)$, and $(-,+,-,-,-)$. (See p. 14 of the lecture notes for the Hebb/Hopfield pattern storage rule.) Are the patterns stable states of the system's dynamics? To what state does the system converge from initial state $(+,-,+,+,+)$?

5. Derive an upper bound on the number of spin flips required for an n-spin SK system with integral interaction coefficients to converge to a stable state under deterministic Glauber dynamics. (*Hint:* Consider the amount of decrease in $H(\sigma)$ per each spin flip.) Express this bound in terms of the size n and number m of binary patterns stored when the system is used as a Hopfield-type associative memory. (Consider the bounds on $H(\sigma)$ that result from using the Hebb/Hopfield pattern storage rule.)

6. Consider a Hopfield network in which m uniformly and independendently generated random n-bit patterns have been stored using the Hebb/Hopfield storage rule. Estimate the probability that a given stored pattern, say $\sigma = (+1, +1, \ldots, +1)$ really is a stable state of the system's dynamics. (*Hint:* Consider the "signal-to-noise" ratio formula (2) on p. 14 of the lecture notes. In an update of the network, each bit of σ is affected by a "signal" term of strength $n - m$, where n is the length of the stored patterns and m is their number, and a roughly normally distributed "noise" term. What is the probability that the "noise" doesn't overcome the "signal" for any of the n bits?) Can you use your probability bound to estimate the maximal number $m(n)$ of randomly generated patterns that can (with high probability) be stably stored in an n-bit Hopfield network?