T-79.250 **Spring 2005**
Combinatorial Models and Stochastic Algorithms
Tutorial 5, February 25
Problems

1. Exhibit a sequence of connected graphs of increasing size and $\Delta = 4$, such that the Gibbs sampler for graph colourings presented on p. 81 of the lecture notes is not irreducible for $q = \Delta + 1 = 5$. (*Hint:* Construct first a "frozen" 5-colouring of the infinite square lattice, i.e. with node set $\mathbf{Z} \times \mathbf{Z}$ and edges connecting Euclidean nearest neighbours. "Frozen" means here that the only transition available from a state is a self-loop with probability 1.)

2. Design a sampler for q-colourings of an arbitrary graph G of maximum degree Δ that is ergodic (regular) when $q \geq \Delta + 1$. (*Hint:* Use transitions based on edge rather than node updates.)

3. Given two subsets A and B of a finite set U, show how to construct an appropriate probability space and two random variables X_A, X_B on U so that:

   (a) $\Pr(X_A = x) = \begin{cases} 1/|A|, & \text{if } x \in A, \\ 0, & \text{if } x \notin A; \end{cases}$

   (b) $\Pr(X_B = x) = \begin{cases} 1/|B|, & \text{if } x \in B, \\ 0, & \text{if } x \notin B; \end{cases}$

   (c) $\Pr(X_A = X_B) = \frac{|A \cap B|}{\max\{|A|,|B|\}}$.

   (This is a technical lemma needed to establish the coupling time bound on the Gibbs sampler for graph colourings on p. 83 of the lecture notes. *Hint:* Consider first some simple concrete example, e.g. $A = \{1, 2, 3, 5, 6\}$, $B = \{1, 2, 3, 4\}$.)

4. Consider the Markov chain with state space $S = \{1, 2\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

   Show that with the update rule

$$s(1, r) = s(2, r) = \begin{cases} 1, & \text{for } r \in [0, \frac{1}{2}) \\ 2, & \text{for } r \in [\frac{1}{2}, 1) \end{cases}$$

   the Propp–Wilson algorithm always terminates on this chain in a single step, whereas with the update rule

$$s(1, r) = \begin{cases} 1, & \text{for } r \in [0, \frac{1}{2}) \\ 2, & \text{for } r \in [\frac{1}{2}, 1) \end{cases} \qquad s(2, r) = \begin{cases} 2, & \text{for } r \in [0, \frac{1}{2}) \\ 1, & \text{for } r \in [\frac{1}{2}, 1) \end{cases}$$

   the algorithm *never* terminates.

5. An appealing "simplification" of the Propp–Wilson algorithm would be to simulate the chains from time $T = 0$ *forward* until coalescense. Show that the samples obtained by this method would not be correctly distributed in the case of the Markov chain with state space $S = \{1, 2\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}.$$