T-79.250 **Spring 2005**
Combinatorial Models and Stochastic Algorithms
Tutorial 4, February 18
Problems

1. Calculate, using the method of canonical paths, an upper bound on the mixing time of the simple random walk on a Boolean hypercube $B_n = \{0,1\}^n$, where at each node u there is a self-loop probability of 1/2, and otherwise a uniform probability $1/2n$ of moving to any of the n nodes at Hamming distance 1 from u.

2. Represent graphically or write out explicitly the transition probability matrix for a Markov chain that has been obtained as a coupling of two copies of the simple two-state Markov chain $\mathcal{M}$ from tutorial problem 2/1. (This chain would be denoted $\mathcal{M}|\mathcal{M}$ in the notation on p. 78 of the lecture notes.) Can you compute the expected value of the coupling time for this chain, when the two component chains are initialised in different states, using the general techniques discussed at tutorial 2 (pp. 51–55 of the lecture notes)?

3. Consider the previous problem more generally: write out explicitly the transition probabilities for a Markov chain that has been obtained as a coupling of two copies of a regular Markov chain $\mathcal{M}$ with transition probability matrix $P = (p_{ij})$. Verify that starting from any initial distribution $\Pr(X_0 = i, Y_0 = j)$, the marginal distributions of the first component chain (and by symmetry also those of the second component chain) remain at all times the same as they would be in an uncoupled $\mathcal{M}$-chain, i.e. that for all $t \geq 0$ and all $i \in S = \{1, \ldots, n\}$,
$$\Pr(X_t = i) = \sum_{j=1}^{n} \Pr(X_t = i, Y_t = j) = (p^{(0)} P^t)_i.$$

4. Establish, using the coupling method, a general geometric bound on the convergence rate of a regular finite Markov chain $\mathcal{M}$, i.e. show that there is a constant $c \in (0,1)$ such that if π is the stationary distribution of $\mathcal{M}$, and $p^{(t)}$ denotes the time t distribution of $\mathcal{M}$, initialised in some arbitrary initial distribution $p^{(0)}$, then $d_{\mathrm{V}}(p^{(t)}, \pi) \leq c^t$.

5. Derive, using the coupling method, rough upper bounds on the mixing times of the random walks discussed in Problem 6 of the previous tutorial and problem 1 of the present one.