T-79.250 **Spring 2005**
Combinatorial Models and Stochastic Algorithms
Tutorial 3, February 11
Problems

1. Denote $C = \{1, \ldots, c\}$, and let π be any probability distribution on the state set $S = C^n$. Prove that the basic Gibbs sampler for π has π as its stationary distribution. (*Hint:* Generalise the argument used in the lecture notes in the case of the Gibbs sampler for the hard-core model.)

2. Consider an arbitrary distribution π on the state set $S = \{0, 1\}^n$. Design for π both (a) a basic Gibbs sampler, and (b) a Metropolis sampler using the Hamming neighbourhood, where S is viewed as a graph whose two nodes are neighbours if and only if their co-ordinate vectors differ in exactly one position. Are the two samplers the same?

3. Verify the claims in Proposition 5.20 of the lecture notes. That is: given a regular reversible Markov chain $\mathcal{M}$ on state set $S = \{1, \ldots, n\}$ with transition matrix P and stationary distribution π, show that the chain $\mathcal{M}'$ with transition matrix $P' = \frac{1}{2}(I_n + P)$ is also regular and reversible, has same stationary distribution π as $\mathcal{M}$, its eigenvalues satisfy $1 = \lambda'_1 > \lambda'_2 \geq \cdots \geq \lambda'_n > 0$, and $\lambda'_{\max} = \lambda'_2 = \frac{1}{2}(1 + \lambda_2)$, where λ_2 is the second largest eigenvalue of $\mathcal{M}$. Estimate the effect of the change from P to P' on the convergence rate of the chain.

4. Determine the exact magnitude of the second eigenvalue λ_2 for the simple symmetric random walk on an n-state cycle discussed on pp. 65 and 66 of the lecture notes, and compare this to the estimates calculated in the notes. (If computing the exact value of λ_2 for general n is overwhelming, try out numerically some small values of n.)

5. Consider a random walk on an undirected graph $G = (V, E)$, where transitions are made from each node u to an adjacent node with uniform probability β/d, where d is the maximum degree of any node in G and $\beta \leq 1$ is a positive constant. In addition, each node u has a self-loop probability of $1 - \beta \deg(u)/d$. Prove that if G is connected and $\beta < 1$, then the corresponding Markov chain $\mathcal{M}_G$ is regular and reversible, with uniform stationary distribution. Moreover, show that the conductance of $\mathcal{M}_G$ is given by the formula

$$\Phi = \beta\mu(G)/d,$$

where $\mu(G)$ is the *edge magnification* (also known as the *isoperimetric number* or *Cheeger constant*) of G, defined as

$$\mu(G) = \min_{0<|U|\leq|V|/2} \frac{|\partial(U)|}{|U|},$$

where $\partial(U) = \{\{u, v\} \in E \mid u \in U, v \notin U\}$.

6. Based on the result of Problem 5, calculate an upper bound on the mixing time of a simple symmetric random walk on an $n \times n$ square lattice with self-loop parameter $0 < 1 - \beta < 1$ and periodic boundary conditions (i.e. each node (i, j), $i, j = 0, \ldots, n-1$, has as neighbours the nodes $(i \pm 1, j \pm 1) \bmod n$).