T-79.250 **Spring 2003**
Combinatorial Models and Stochastic Algorithms
Tutorial 9, March 28
Problems

1. Verify by direct calculation that in the simulated annealing algorithm the finite-temperature Gibbsian distributions $\pi^{(T)}$ for $T > 0$ do indeed converge pointwise to the desired limit distribution π^* as $T \to 0$.

2. Apply Dobrushin's convergence rate bound (Lemma 6.2 of the lecture notes) to the circular random walk on n states (n even), discussed on pp. 85 and 88 of the notes, and also in Problem 6/3 of the tutorials. (Note that in order to get nontrivial results, you probably need to apply Dobrushin's bound to some power P^d of the transition matrix P, rather than to the matrix itself.)

3. Consider the nonhomogeneous Markov chain on the state space $S = \{1, 2\}$ with transition probability matrices given for $n \geq 1$ by:

$$P^{(2n)} = \begin{pmatrix} 1/2n & 1-1/2n \\ 1/2n & 1-1/2n \end{pmatrix}, \quad P^{(2n+1)} = \begin{pmatrix} 1-1/(2n+1) & 1/(2n+1) \\ 1-1/(2n+1) & 1/(2n+1) \end{pmatrix}.$$

   For a given initial distribution μ and time $m \geq 1$, compute explicit expressions for the distributions $\mu^T P(m,k)$, $k \geq 1$. Show that the chain is weakly but not strongly ergodic.

4. Consider a simple state space graph with states $S = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ and neighbourhood structure $N(\sigma_i) = \{\sigma_{i-1}, \sigma_{i+1}\}$, where the indices are computed modulo 4. Write down explicitly the transition probability matrix of the simulated annealing algorithm at temperature t for this system, when the function to be minimised is given by $H(\sigma_0) = 1$, $H(\sigma_1) = 2$, $H(\sigma_2) = 0$, $H(\sigma_3) = 2$. Given a cooling schedule where the temperature at step k of the algorithm is $t_k > 0$, what is the probability that the algorithm when initialised in the locally optimal state σ_0 will stay there forever? Find a sequence t_k for which this probability is nonzero. What kind of cooling schedule would, according to Theorem 6.5 (p. 118 of the notes) guarantee asymptotic convergence to the globally optimal state σ_2?

5. The NP-complete PARTITION problem is defined as follows: given a sequence of $2n$ nonnegative integers $x_1, \ldots, x_{2n}$, is there a subsequence of n numbers whose sum is exactly half the sum of the whole sequence? Formulate the task of finding approximate partitions of an integer sequence as a minimisation problem, and present a simulated annealing approach to solving it. What kinds of cooling schedules would Theorem 6.5 suggest for your algorithm in the case of input sequences consisting of $2n$ numbers from the interval $[0, N]$? (You might consider also actually implementing your algorithm and experimenting with some more realistic cooling schedules.)