T-79.250 **Spring 2003**
Combinatorial Models and Stochastic Algorithms
Tutorial 8, March 21
Problems

1. Given two subsets A and B of a finite set U, show how to construct an appropriate probability space and two random variables X_A, X_B on U so that:

   (a) $\Pr(X_A = x) = \begin{cases} 1/|A|, & \text{if } x \in A, \\ 0, & \text{if } x \notin A; \end{cases}$

   (b) $\Pr(X_B = x) = \begin{cases} 1/|B|, & \text{if } x \in B, \\ 0, & \text{if } x \notin B; \end{cases}$

   (c) $\Pr(X_A = X_B) = \frac{|A \cap B|}{\max\{|A|,|B|\}}$.

   (This is a technical lemma needed to establish the coupling time bound on the Markov chain for colouring low-degree graphs on p. 105 of the lecture notes.)

2. Consider the Markov chain with state space $S = \{1, 2\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

   Show that with the update rule

$$s(1,r) = s(2,r) = \begin{cases} 1, & \text{for } r \in [0, \frac{1}{2}) \\ 2, & \text{for } r \in [\frac{1}{2}, 1) \end{cases}$$

   the Propp–Wilson algorithm always terminates on this chain in a single step, whereas with the update rule

$$s(1,r) = \begin{cases} 1, & \text{for } r \in [0, \frac{1}{2}) \\ 2, & \text{for } r \in [\frac{1}{2}, 1) \end{cases} \qquad s(2,r) = \begin{cases} 2, & \text{for } r \in [0, \frac{1}{2}) \\ 1, & \text{for } r \in [\frac{1}{2}, 1) \end{cases}$$

   the algorithm *never* terminates.

3. An appealing "simplification" of the Propp–Wilson algorithm would be to simulate the chains from time $T = 0$ *forward* until coalescense. Show that the samples obtained by this method would not be correctly distributed in the case of the Markov chain with state space $S = \{1, 2\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}.$$

4. Consider a Propp–Wilson simulation of the Gibbs sampler for an n-spin ferromagnetic Ising spin system (or more generally a ferromagnetic SK spin system) at inverse temperature β. Show that the update rule $s(x, r)$ introduced on p. 108 of the lecture notes is in this case congruent with the partial order $\sqsubseteq$ on the system configurations $\sigma, \sigma' \in \{-1, 1\}^n$ defined as:

$$\sigma \sqsubseteq \sigma' \quad \text{if} \quad \sigma_i \le \sigma'_i \quad \text{for all } i = 1, \ldots, n.$$

(PLEASE TURN OVER)

That is, show that for any pair of states $\sigma, \sigma' \in \{-1, 1\}^n$, any choice of site i to be updated, and any value of $r \in [0, 1)$:

$$\sigma \sqsubseteq \sigma' \quad \text{implies} \quad s(\sigma, \langle i, r\rangle) \sqsubseteq s(\sigma', \langle i, r\rangle),$$

where $s(\sigma, \langle i, r\rangle)$ denotes the system state obtained from state σ by updating site i as determined by the rule $s(\sigma_i, r)$ (and respectively for $s(\sigma', \langle i, r\rangle)$). [The significance of this result is of course that it permits the use of the monotone, or "sandwiching" version of the Propp–Wilson algorithm, which is exponentially more efficient than the basic method.]

5. [*Noncredit problem.*] Estimate the success probability of the card trick presented in class. (You probably would want to use some computerised tools for this.)