T-79.250 **Spring 2003**
Combinatorial Models and Stochastic Algorithms
Tutorial 6, March 7
Problems

1. Let $\mathcal{M}$ be a regular Markov chain on state set $S = \{1, \dots, n\}$, with transition probability matrix P and stationary distribution π. Verify the claim that $\mathcal{M}$ is reversible if and only if the matrix $A = D^{1/2} P D^{-1/2}$ is symmetric, where $D^{1/2}$ is the diagonal matrix $\mathrm{diag}(\sqrt{\pi_1}, \dots, \sqrt{\pi_n})$.

2. Verify the claims in Proposition 5.20 of the lecture notes, viz. that under the assumptions of Problem 1, the chain $\mathcal{M}'$ with transition matrix $P' = \frac{1}{2}(I_n + P)$ is also regular and reversible, has same stationary distribution π as $\mathcal{M}$, its eigenvalues satisfy $1 = \lambda'_1 > \lambda'_2 \geq \cdots \geq \lambda'_n > 0$, and $\lambda'_{\max} = \lambda'_2 = \frac{1}{2}(1 + \lambda_2)$. Estimate the effect of the change from P to P' on the convergence rate of the chain.

3. Determine the exact magnitude of the second eigenvalue λ_2 for the circular Markov chain on n states discussed on pp. 85 and 88 of the lecture notes, and compare this to the estimates calculated in the notes. (If computing the exact value of λ_2 for general n is overwhelming, try out numerically some small values of n.)

4. Consider a random walk on an undirected graph $G = (V, E)$, where transitions are made from each node u to an adjacent node with uniform probability β/d, where d is the maximum degree of any node in G and $\beta \leq 1$ is a positive constant. In addition, each node u has a self-loop probability of $1 - \beta \deg(u)/d$. Prove that if G is connected and $\beta < 1$, then the corresponding Markov chain $\mathcal{M}_G$ is regular and reversible, with uniform stationary distribution. Moreover, show that the conductance of $\mathcal{M}_G$ is given by the formula
$$\Phi = \beta \mu(G)/d,$$
where $\mu(G)$ is the *edge magnification* (also known as the *isoperimetric number* or *Cheeger constant*) of G, defined as
$$\mu(G) = \min_{0 < |U| \leq |V|/2} \frac{|\partial(U)|}{|U|},$$
where $\partial(U) = \{\{u, v\} \in E \mid u \in U, v \notin U\}$.

5. Based on the result of Problem 4, calculate an upper bound on the mixing time of a uniform random walk on an $n \times n$ lattice with self-loop parameter $0 < 1 - \beta < 1$ and periodic boundary conditions (i.e. each node (i, j), $i, j = 0, \dots, n-1$, has as neighbours the nodes $(i \pm 1, j \pm 1) \bmod n$).

6. Calculate, using the method of canonical paths, an upper bound on the mixing time of the simple random walk on a Boolean hypercube $B_n = \{0, 1\}^n$, where at each node u there is a self-loop probability of $1/2$, and otherwise a uniform probability $1/2n$ of moving to any of the n nodes at Hamming distance 1 from u.