T-79.250 **Spring 2003**

Combinatorial Models and Stochastic Algorithms

Tutorial 5, February 28

Problems

1. Let $0 < p, q < 1$. Consider the Markov chain on state set $\{1, 2\}$ given by transition probability matrix
$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}.$$
Compute explicit expressions for the k-step transition probability matrices P^k, $k \geq 1$, and their (elementwise) limit $P^\infty = \lim_{k\to\infty} P^k$. Observe that if ρ is any distribution vector, then $\rho P^\infty = \pi$, where π is the stationary distribution of the chain.

2. Given the transition probability matrix P of an irreducible Markov chain on state set $S = \{1, \ldots, n\}$, show how to obtain from P the expected hitting times μ_{i1} for initial states $i \neq 1$. Use this result to compute the expected time to reach state 0 from each of the other states in a circular Markov chain on set $\{0, 1, 2, 3\}$ that at each step moves from state i to state $i \pm 1 \pmod 4$ with probability 1/2.

3. Consider Problem 3(a) from the previous tutorial, i.e. the Markov chain determined by a king making random moves on a chessboard. You presumably already showed that this chain is irreducible and aperiodic, and hence has a unique stationary distribution π. Determine π. (*Hint:* View the chessboard as a graph.)

4. Denote $C = \{1, \ldots, c\}$, and let π be any probability distribution on the state set $S = C^n$. Prove that the basic Gibbs sampler for π has π as its stationary distribution. (*Hint:* Generalise the argument used in the lecture notes in the case of the Gibbs sampler for the hard-core model.)

5. Consider an arbitrary distribution π on the state set $S = \{0, 1\}^n$. Design for π both (a) a basic Gibbs sampler, and (b) a Metropolis sampler using the Hamming neighbourhood, where S is viewed as a graph whose two nodes are neighbours if and only if their co-ordinate vectors differ in exactly one position. Are the two samplers different?

6. Design a Metropolis sampler for the Gibbs-Boltzmann distribution of the SK spin glass model at temperature $\beta = 1/kT$, using the same Hamming neighbourhood structure as in the previous problem. (*NB:* In the neural networks literature, the resulting chain is called the "Boltzmann machine.")