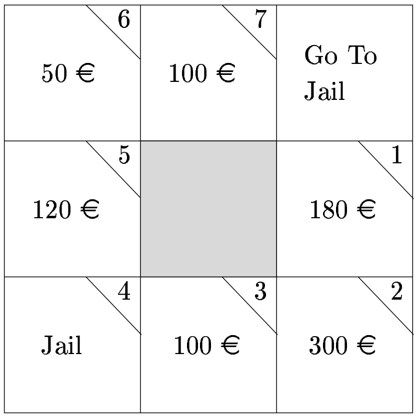
T-79.250 **Spring 2003**

Combinatorial Models and Stochastic Algorithms

Tutorial 4, February 21

Problems

1. Consider a game of Monopoly played on the following simplified board:



Assume that the game is played with a coin instead of dice, so that when the coin turns up "heads" the player advances one square, and with "tails" the player advances two squares. Compute the stationary distribution of square occupancies in an infinitely long game, and based on this, the relative rent rate obtained from each square.

2. Prove that any two communicating states in a Markov chain have the same period. Observe as a corollary that if a Markov chain is irreducible and has a state i such that $p_{ii} > 0$, then it is also aperiodic.

3. (a) Consider a chessboard with a lone king making random moves, meaning that at each move, the king chooses one of its permissible next-state squares uniformly at random. Is the corresponding Markov chain irreducible and/or aperiodic?
   (b) Same question, but with the king replaced by a bishop.
   (c) Same question, but now with a knight.

4. Consider a finite irreducible Markov chain with transition matrix P that has period $d > 1$. Show that the corresponding Markov chain with transition matrix P^d is aperiodic, but no longer irreducible. What can you learn about the periodicity structure of the chain determined by P by looking at the minimal closed sets ("communicating components") of the chain determined by P^d?

5. Given a Markov chain $(X_0, X_1, \dots)$ on a state space $S = \{1, \dots, n\}$, define the following family of *hitting time* random variables $T_j^{(i)}$ for $i, j \in S$:

$$T_j^{(i)} = \begin{cases} \min\{t \geq 1 \mid X_t = j\}, & \text{if } X_0 = i, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Then one can define the *expected hitting time* for states $i, j \in S$ as $\mu_{ij} = E[T_j^{(i)} \mid X_0 = i]$, and the *expected return time* for a state $i \in S$ as $\mu_i = \mu_{ii}$. Furthermore, one can define a notion of "expected number of visits to j before first return to i" as:

$$\rho_j^{(i)} = \sum_{t \geq 0} \Pr(X_t = j, T_j^{(i)} > t \mid X_0 = i).$$

(Note that by definition $\rho_i^{(i)} = 1$ for any $i \in S$.) Prove, using these notions, that for an irreducible aperiodic Markov chain on $S = \{1, \ldots, n\}$, the unique stationary distribution π can be characterised as

$$\pi = \frac{1}{\mu_1}(1, \rho_2^{(1)}, \ldots, \rho_n^{(1)}),$$

and deduce by symmetry that this is in fact the same as

$$\pi = (\frac{1}{\mu_1}, \ldots, \frac{1}{\mu_n}).$$

(I.e., what needs to be shown is that the given vector represents a distribution and is stationary w.r.t. the chain's transition matrix. Since it is known that the stationary distribution is unique, this establishes the result.)

6. Given a probability distribution p on the state space $S = \{1, \ldots, n\}$, denote the probability mass of any $A \subseteq S$ by $p(A) = \sum_{i \in A} p_i$. Then the *total variation distance* between two probability distributions p and q on S is defined as

$$d_{\mathrm{TV}}(p, q) = \max_{A \subseteq S} |p(A) - q(A)|.$$

Establish the following simple characterisation for this distance measure:

$$d_{\mathrm{TV}}(p, q) = \frac{1}{2} \sum_{i=1}^{n} |p_i - q_i|,$$

i.e., $d_{\mathrm{TV}}(p, q)$ is just the L_1-distance between the stochastic n-vectors p and q, normalised to the interval $[0, 1]$. (*Hint:* For given p and q, consider the set $A = \{i \in S \mid p_i \geq q_i\}$, and its complement.) What are the relationships of total variation distance to the pointwise maximum distance measure, $\|p - q\|_\infty = \max_i |p_i - q_i|$, and to the Euclidean distance measure, $\|p - q\|_2 = \left(\sum_i (p_i - q_i)^2\right)^{1/2}$?