T-79.250 **Spring 2003**

Combinatorial Models and Stochastic Algorithms

Tutorial 3, February 14

Problems

1. Let X be the number of cycles in a random graph $G \in \mathcal{G}(n,p)$, where $p = c/n$. Find an exact formula for $E[X]$, and estimate the asymptotics of $E[X]$ when (a) $c < 1$ and (b) $c = 1$.

2. Consider the space Ω_n of random equiprobable permutations of $[n] = \{1,\dots,n\}$. A permutation $\pi \in \Omega_n$ contains an *increasing subsequence of length* k, if there are indices $i_1 < \cdots < i_k$ such that $\pi(i_1) < \cdots < \pi(i_k)$.

   (a) Show that almost no permutation $\pi \in \Omega_n$ contains an increasing subsequence of length at least $e\sqrt{n}$. (*Hint:* Let $I_k(\pi)$ be the number of increasing subsequences of length k contained in π. Estimate $E[I_k]$.)

   (b) Denote the length of a maximal increasing subsequence contained in a permutation π by $I(\pi)$, and correspondingly the length of a maximal *decreasing* subsequence by $D(\pi)$. Erdős and Szekeres proved in 1935 that $I(\pi)D(\pi) \geq n$ for any permutation π of $[n]$, (The proof is simple & elegant; think about it or look it up in any combinatorics textbook under "Ramsey theory.") Deduce from this result and the result of part (a) that almost every permutation $\pi \in \Omega_n$ contains an increasing subsequence of length at least $\sqrt{n}/e$.

3. Compute the expected value of the clustering coefficient $\mathcal{C}(G)$ for an ER random graph $G \in \mathcal{G}(n,p)$. Give also some estimates for the expected value of the characteristic path length $\mathcal{L}(G)$.

4. Compute the clustering coefficient $\mathcal{C}(G)$, characteristic path length $\mathcal{L}(G)$, and distribution of node degrees for a circulant graph C_{nk}. (It suffices to compute these quantities asymptotically for fixed k and large n.) What is the edge density $p = e(C_{nk})/\binom{n}{2}$ for such a graph? What is the effect on $\mathcal{L}(G)$ of a single randomly added shortcut edge?

5. Compute the clustering coefficient $\mathcal{C}(G)$, characteristic path length $\mathcal{L}(G)$, and distribution of node degrees for a "caveman graph" consisting of k "caves" of r nodes each. What is the edge density p for such a graph? (Recall that a "caveman graph" is a cyclic arrangement of k appropriately modified r-cliques. It suffices to compute these quantities asymptotically for (a) fixed r and large k and (b) the case $r \sim k \sim \sqrt{n}$ for large n.)