1. Let $X$ be the number of cycles in a random graph $G \in \mathcal{G}(n, p)$, where $p = c/n$. Find an exact formula for $E[X]$, and estimate the asymptotics of $E[X]$ when (a) $c < 1$ and (b) $c = 1$.

2. Consider the space $\Omega_n$ of random equiprobable permutations of $[n] = \{1, \ldots, n\}$. A permutation $\pi \in \Omega_n$ contains an increasing subsequence of length $k$, if there are indices $i_1 < \cdots < i_k$ such that $\pi(i_1) < \cdots < \pi(i_k)$.

   (a) Show that almost no permutation $\pi \in \Omega_n$ contains an increasing subsequence of length at least $e\sqrt{n}$. (Hint: Let $I_k(\pi)$ be the number of increasing subsequences of length $k$ contained in $\pi$. Estimate $E[I_k]$.)

   (b) Denote the length of a maximal increasing subsequence contained in a permutation $\pi$ by $I(\pi)$, and correspondingly the length of a maximal decreasing subsequence by $D(\pi)$. Erdős and Szekeres proved in 1935 that $I(\pi)D(\pi) \geq n$ for any permutation $\pi$ of $[n]$. (The proof is simple & elegant; think about it or look it up in any combinatorics textbook under “Ramsey theory.”) Deduce from this result and the result of part (a) that almost every permutation $\pi \in \Omega_n$ contains an increasing subsequence of length at least $\sqrt{n/e}$.

3. Compute the expected value of the clustering coefficient $C(G)$ for an ER random graph $G \in \mathcal{G}(n, p)$. Give also some estimates for the expected value of the characteristic path length $L(G)$.

4. Compute the clustering coefficient $C(G)$, characteristic path length $L(G)$, and distribution of node degrees for a circulant graph $C_{nk}$. (It suffices to compute these quantities asymptotically for fixed $k$ and large $n$.) What is the edge density $p = e(C_{nk})/(\binom{n}{2})$ for such a graph? What is the effect on $L(G)$ of a single randomly added shortcut edge?

5. Compute the clustering coefficient $C(G)$, characteristic path length $L(G)$, and distribution of node degrees for a “caveman graph” consisting of $k$ “caves” of $r$ nodes each. What is the edge density $p$ for such a graph? (Recall that a “caveman graph” is a cyclic arrangement of $k$ appropriately modified $r$-cliques. It suffices to compute these quantities asymptotically for (a) fixed $r$ and large $k$ and (b) the case $r \sim k \sim \sqrt{n}$ for large $n$.)