T-79.250 **Spring 2003**
Combinatorial Models and Stochastic Algorithms
Tutorial 10, April 4
Problems

1. The *permanent* of an $n \times n$ matrix $A = (a_{i,j})$ is defined as

$$\mathrm{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)},$$

where S_n denotes the group of permutations of the set $\{1, \ldots, n\}$. Consider for example the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Interpret this A as the adjacency matrix of a $(4+4)$-node bipartite graph G_A, compute $\mathrm{per}(A)$ and observe that it is the same as the number of perfect matchings in G_A. (A *perfect matching* in a graph is a subset of its edges such that each node is incident with exactly one of the chosen edges.) Explain why this correspondence holds for any binary-valued matrix $A \in \{0,1\}^{n \times n}$, and show that the problem of computing permanents of such matrices is self-reducible in an appropriate sense.

2. Consider a simple self-reduction setting for an NP relation R, where for any input x of length $|x| > n_0$, the set of witnesses $R(x) = \{w \mid R(x, w)\}$ can be partitioned into two disjoint classes by polynomially computable length-decreasing self-reduction functions f_0 and f_1, i.e. for $|x| > n_0$,

$$R(x) = R(f_0(x)) \uplus R(f_1(x)), \quad |f_0(x)|, |f_1(x)| < |x|.$$

Assume the availability of a perfect small-scale sampler $U_R(x)$ for generating elements $w \in R(x)$ uniformly at random for inputs x of length $|x| \le n_0$, and an FPRAS $A(x, \epsilon)$ for approximately counting the number of elements in $R(x)$ for $|x| > n_0$. Show how these can be combined to obtain an FPAUS $S(x, \delta)$ for sampling elements in $R(x)$ almost uniformly at random for arbitrary inputs x.

3. Continuing the previous problem setting, assume conversely the availability of a perfect small-scale witness-counter $N_R(x)$ for computing the size of $R(x)$ for $|x| \le n_0$, and an FPAUS $S(x, \delta)$ for sampling elements in $R(x)$ almost uniformly at random for $|x| > n_0$. Show how these can be combined to obtain an FPRAS $A(x, \epsilon)$ for approximately counting the number of elements in $R(x)$ for arbitrary inputs x.

(PLEASE TURN OVER)

4. (a) Let $A_1, A_2, \ldots$ be a collection of events, and $A = \cap_{n\geq 1} \cup_{m\geq n} A_m$ the event that infinitely many of the A_m occur. Prove the "first Borel-Cantelli lemma", which states that if $\sum_{n\geq 1} \Pr(A_n) < \infty$, then $\Pr(A) = 0$. (*Hint:* $A \subseteq \cup_{m\geq n} A_m$ for all $m \geq 1$.)

   (b) Based on the previous result, prove the following special case of Kolmogorov's Strong Law of Large Numbers: for any sequence $X_1, X_2, \ldots$ of i.i.d. random variables for which $E(X_1) = 0$ and $E(X_1^4) < \infty$, $\frac{1}{n}(X_1 + X_2 + \cdots + X_n) \to 0$ almost surely, i.e. denoting $S_n = \sum_{k=1}^{n} X_k$,

$$\Pr(\exists \epsilon > 0 \text{ s.th. } |S_n|/n > \epsilon \text{ infinitely often}) = 0.$$

   (*Hint:* For a given $\epsilon > 0$, consider the events $A_n = \{|S_n| > n\epsilon\} = \{S_n^4 > (n\epsilon)^4\}$. Apply Markov's inequality and the fact that for independent random variables X_1 and X_2, $E(X_1 X_2) = E(X_1)E(X_2)$.)

5. Let $\mathcal{M}$ be a regular finite Markov chain with state space S and stationary distribution π. Recall from tutorial problem 4/5 that for any $i \in S$, $\pi_i = 1/\mu_i$, where μ_i is the expected return time to i. Let then $A \subseteq S$ be any set of states of $\mathcal{M}$, and denote by τ_k, $k \geq 1$, the sequence of return times to A in a sample path ("run") of $\mathcal{M}$. Show that given any initial distribution μ for $\mathcal{M}$, the condition

$$\lim_{k\to\infty} \frac{\tau_k}{k} = \frac{1}{\sum_{i\in A} \pi_i}$$

holds μ-almost surely.