1. The permanent of an \( n \times n \) matrix \( A = (a_{i,j}) \) is defined as

\[
\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)},
\]

where \( S_n \) denotes the group of permutations of the set \( \{1, \ldots, n\} \). Consider for example the matrix

\[
A = \begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}.
\]

Interpret this \( A \) as the adjacency matrix of a \((4 + 4)\)-node bipartite graph \( G_A \), compute \( \text{per}(A) \) and observe that it is the same as the number of perfect matchings in \( G_A \). (A perfect matching in a graph is a subset of its edges such that each node is incident with exactly one of the chosen edges.) Explain why this correspondence holds for any binary-valued matrix \( A \in \{0, 1\}^{n \times n} \), and show that the problem of computing permanents of such matrices is self-reducible in an appropriate sense.

2. Consider a simple self-reduction setting for an NP relation \( R \), where for any input \( x \) of length \( |x| > n_0 \), the set of witnesses \( R(x) = \{w \mid R(x, w)\} \) can be partitioned into two disjoint classes by polynomially computable length-decreasing self-reduction functions \( f_0 \) and \( f_1 \), i.e. for \( |x| > n_0 \),

\[
R(x) = R(f_0(x)) \cup R(f_1(x)), \quad |f_0(x)|, |f_1(x)| < |x|.
\]

Assume the availability of a perfect small-scale sampler \( U_R(x) \) for generating elements \( w \in R(x) \) uniformly at random for inputs \( x \) of length \( |x| \leq n_0 \), and an FPRAS \( A(x, \epsilon) \) for approximately counting the number of elements in \( R(x) \) for \( |x| > n_0 \). Show how these can be combined to obtain an FPAUS \( S(x, \delta) \) for sampling elements in \( R(x) \) almost uniformly at random for arbitrary inputs \( x \).

3. Continuing the previous problem setting, assume conversely the availability of a perfect small-scale witness-counter \( N_R(x) \) for computing the size of \( R(x) \) for \( |x| \leq n_0 \), and an FPAUS \( S(x, \delta) \) for sampling elements in \( R(x) \) almost uniformly at random for \( |x| > n_0 \). Show how these can be combined to obtain an FPRAS \( A(x, \epsilon) \) for approximately counting the number of elements in \( R(x) \) for arbitrary inputs \( x \).

(PLEASE TURN OVER)
4. (a) Let $A_1, A_2, \ldots$ be a collection of events, and $A = \bigcap_{n \geq 1} \bigcup_{m \geq n} A_m$ the event that infinitely many of the $A_m$ occur. Prove the “first Borel-Cantelli lemma”, which states that if $\sum_{n \geq 1} \Pr(A_n) < \infty$, then $\Pr(A) = 0$. (Hint: $A \subseteq \bigcup_{m \geq n} A_m$ for all $m \geq 1$.)

(b) Based on the previous result, prove the following special case of Kolmogorov’s Strong Law of Large Numbers: for any sequence $X_1, X_2, \ldots$ of i.i.d. random variables for which $E(X_1) = 0$ and $E(X_1^4) < \infty$, $rac{1}{n}(X_1 + X_2 + \cdots + X_n) \to 0$ almost surely, i.e. denoting $S_n = \sum_{k=1}^n X_k$,

$$\Pr(\exists \epsilon > 0 \text{ s.th. } |S_n|/n > \epsilon \text{ infinitely often}) = 0.$$  

(Hint: For a given $\epsilon > 0$, consider the events $A_n = \{|S_n| > n\epsilon\} = \{S_n^4 > (n\epsilon)^4\}$. Apply Markov’s inequality and the fact that for independent random variables $X_1$ and $X_2$, $E(X_1X_2) = E(X_1)E(X_2)$.)

5. Let $\mathcal{M}$ be a regular finite Markov chain with state space $S$ and stationary distribution $\pi$. Recall from tutorial problem 4/5 that for any $i \in S$, $\pi_i = 1/\mu_i$, where $\mu_i$ is the expected return time to $i$. Let then $A \subseteq S$ be any set of states of $\mathcal{M}$, and denote by $\tau_k$, $k \geq 1$, the sequence of return times to $A$ in a sample path (“run”) of $\mathcal{M}$. Show that given any initial distribution $\mu$ for $\mathcal{M}$, the condition

$$\lim_{k \to \infty} \frac{\tau_k}{k} = \frac{1}{\sum_{i \in A} \pi_i}$$

holds $\mu$-almost surely.