1. Show that in a canonical ensemble of microstates (i.e. w.r.t. the Gibbs probability distribution), one obtains

\[ \langle H^2 \rangle = 1 \frac{\partial^2 Z}{Z \partial \beta^2} \]

and, using this result, that the specific heat of the system,

\[ C_V = \frac{\partial \langle H \rangle}{\partial T}, \]

satisfies

\[ C_V = \frac{\beta}{T} [\langle H^2 \rangle - \langle H \rangle^2], \]

i.e. indicates the fluctuations around the average energy value.

2. Compute the thermodynamic potential (average energy), specific heat, and entropy of a decoupled (\( J = 0 \)) Ising system with \( N \) spins at temperature \( T \) and external field \( h \).

3. Same as problems 1 and 2, but with respect to magnetic susceptibility

\[ \chi = \frac{\partial \langle M \rangle}{\partial h} = \beta [\langle M^2 \rangle - \langle M \rangle^2] \]

in place of specific heat.

4. Derive an upper bound on the number of spin flips required for an \( N \)-spin SK system to converge to a metastable state under deterministic Glauber dynamics. Express this bound in terms of the size \( N \) and number \( M \) of binary patterns stored when the system is used as a Hopfield-type associative memory. (See p. 16 of the lecture notes or any reference work on neural networks for the Hebb/Hopfield pattern storage rule.)

5. Construct a 5-bit Hopfield associative memory network for the patterns (+, +, +, +, +), (+, -, -, +, -), and (-, +, -, -). (See p. 16 of the lecture notes or any reference work on neural networks for the Hebb/Hopfield pattern storage rule.) Are the patterns stable states of the system’s dynamics? To what state does the system converge from initial state (+, -, +, +, +)?

6. Estimate how many randomly generated bit patterns, where each bit is chosen to be +1 or -1 with equal probability and independently of all the other bits, can be stored in an \( N \)-bit Hopfield associative memory network using the standard Hebb/Hopfield storage rule, so that the patterns with high probability become stable states of the resulting system’s dynamics. (You may want to consult the neural networks literature for this problem.)