# TURING MACHINES

- Turing machines are used as the formal model of algorithms.
- Turing machines can simulate arbitrary algorithms with inconsequential loss of efficiency using a single data structure: a string of symbols.

**Definition.** A Turing machine is a quadruple $M = (K, \Sigma, \delta, s)$:
$K$: a finite set of *states*
$\Sigma$: a finite set of symbols (*alphabet* of $M$) with $\sqcup, \triangleright \in \Sigma$
$\delta$: $K \times \Sigma \mapsto (K \cup \{\mathrm{h}, \text{“yes”}, \text{“no”}\}) \times \Sigma \times \{\rightarrow, \leftarrow, -\}$
(*transition function* of $M$)
h: halting state; “yes”: accepting state; “no”: rejecting state
cursor directions: $\rightarrow$ (move right), $\leftarrow$ (move left), $-$ (stay)

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio



**Example** Turing machine $M = (K, \Sigma, \delta, s)$:

$K = \{s, q\}$
$\Sigma = \{0, 1, \sqcup, \triangleright\}$

$\delta$ :

| $p \in K$ | $\sigma \in \Sigma$ | $\delta(p, \sigma)$ |
|---|---|---|
| $s$, | $0$ | $(s, 0, \rightarrow)$ |
| $s$, | $1$ | $(s, 1, \rightarrow)$ |
| $s$, | $\sqcup$ | $(q, \sqcup, \leftarrow)$ |
| $s$, | $\triangleright$ | $(s, \triangleright, \rightarrow)$ |
| $q$, | $0$ | $(h, 1, -)$ |
| $q$, | $1$ | $(q, 0, \leftarrow)$ |
| $q$, | $\sqcup$ | $(q, \sqcup, -)$ |
| $q$, | $\triangleright$ | $(h, \triangleright, \rightarrow)$ |

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio



- Function $\delta$ is the “program” of the machine.
- For current state $q \in K$ and current symbol $\sigma \in \Sigma$,
  $\delta(q, \sigma) = (p, \rho, D)$ where $p$ is the new state,
  $\rho$ is the symbol to be overwritten on $\sigma$ and
  $D \in \{\rightarrow, \leftarrow, -\}$ is the direction in which the cursor will move.
- For any states $p, q$, $\delta(q, \triangleright) = (p, \rho, D)$ with $\rho = \triangleright$ and $D = \rightarrow$.
- If the machine moves off the right end of the string, it reads $\sqcup$ (the string becomes longer but it cannot become shorter; keeps track of the space used).

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio



- The program starts with (i) initial state $s$,
  (ii) the string initialized to $\triangleright x$ where $x$ is a finitely long string in $(\Sigma - \{\sqcup\})^*$ ($x$ is the *input* of the machine) and
  (iii) the cursor pointing to $\triangleright$.
- A machine has *halted* iff one of the 3 halting states $(\mathrm{h}, \text{“yes”}, \text{“no”})$ has been reached.
- If “yes” has been reached, the machine *accepts* the input.
  If “no” has been reached, the machine *rejects* the input.
- *Output* $M(x)$ of a machine $M$ on input $x$:
  (i) If $M$ accepts/rejects, then $M(x) =$ “yes”/“no”.
  (ii) If h has been reached, $M(x) = y$
  where $\triangleright y \sqcup \sqcup \ldots$ is the string of $M$ at the time of halting.
  (iii) If $M$ never halts on input $x$, then $M(x) = \nearrow$

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio

## A formal model of the operation

- A configuration $(q, w, u)$:
  $q \in K$ is the current state and $w, u \in \Sigma^*$ where
  (i) $w$ is the string to the left of the cursor including the symbol scanned by the cursor and
  (ii) $u$ is the string to the right of the cursor.
- The relation $\stackrel{M}{\to}$ (yields in one step): $(q, w, u) \stackrel{M}{\to} (q', w', u')$
  Let $\sigma$ be the last symbol of $w$ and $\delta(q, \sigma) = (p, \rho, D)$.
  Then $q' = p$, and $w', u'$ are obtained according to $(p, \rho, D)$.
  For example, if $D = \rightarrow$, then
  (i) $w'$ is $w$ with its last symbol replaced by $\rho$ and the first symbol of $u$ appended to it ($\sqcup$ if $u$ is empty) and
  (ii) $u'$ is $u$ with the first removed (or empty, if $u$ is empty).

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio



- Yields in k steps: $(q, w, u) \stackrel{M}{\to}^k (q', w', u')$
  iff there are configurations $(q_i, w_i, u_i), i = 1, \ldots, k+1$ such that
  $(q, w, u) = (q_1, w_1, u_1)$
  $(q_i, w_i, u_i) \stackrel{M}{\to} (q_{i+1}, w_{i+1}, u_{i+1}), i = 1, \ldots, k$
  $(q', w', u') = (q_{k+1}, w_{k+1}, u_{k+1})$
- Yields: $(q, w, u) \stackrel{M}{\to}^* (q', w', u')$
  iff there is some $k \geq 0$ such that $(q, w, u) \stackrel{M}{\to}^k (q', w', u')$.
  (Transitive and reflexive closure of $\stackrel{M}{\to}$.)

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio



## Turing machines as algorithms

Turing machines are natural for solving problems on strings:

- Let $L \subset (\Sigma - \{\sqcup\})^*$ be a language.
  A Turing machine $M$ *decides* $L$ iff for every string $x \in (\Sigma - \{\sqcup\})^*$, if $x \in L, M(x) =$ "yes" and if $x \notin L, M(x) =$ "no" .
- If $L$ is decided by a Turing machine, $L$ is a *recursive* language.
- A Turing machine $M$ *accepts* $L$ iff for every string $x \in (\Sigma - \{\sqcup\})^*$, if $x \in L$, then $M(x) =$ "yes" but if $x \notin L, M(x) = \nearrow$ .

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio



- If $L$ is accepted by some Turing machine, $L$ is a *recursively enumerable* language.
  **Proposition.** If $L$ is recursive, then it is recursively enumerable.
- String functions $f : (\Sigma - \{\sqcup\})^* \mapsto \Sigma^*$
  A Turing machine $M$ *computes* $f$ iff for every string $x \in (\Sigma - \{\sqcup\})^*$, $M(x) = f(x)$.
- If such an $M$ exists, $f$ is called a *recursive function*.

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio

## Solving problems using Turing machines

- Instances of the problem need to be represented by strings.
- Solving a decision problem amounts to deciding the language consisting of the encodings of the "yes" instances of the problem.
- An optimization problem is solved by a Turing machine that computes the appropriate function from strings to strings (where the output is similarly represented as a string).

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio

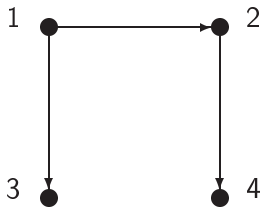


## How does representation affect solvability?

- Any "finite" mathematical object can be represented by a finite string over an appropriate alphabet.

**Example.**

Graph:




Representations as a string:

"$\{(1, 10), (1, 11), (10, 100)\}$"

"$(0110, 0001, 0000, 0000)$"

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio



## Representation vs. solvability?

- All acceptable encodings are related polynomially:

  If A and B are both "reasonable" representations of the same set of instances, and representation A of an instance is a string with $n$ symbols, the representation B of the same instance has length at most $p(n)$ for some polynomial $p$.
- Exception: unary representation of numbers requires exponentially more symbols than the binary representation.
- Reasonable succinct input representation is assumed.

  In particular, numbers are always represented in binary.

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio



## Turing machines with multiple strings

- Turing machines with multiple strings and associated cursors are more convenient.
- They can be simulated by an ordinary Turing machine with an inconsequential loss of efficiency.
- A $k$-string Turing machine, where $k \geq 1$ is an integer, is a quadruple $M = (K, \Sigma, \delta, s)$ where $\delta$ has been generalized:
  $\delta$: $K \times \Sigma^k \mapsto (K \cup \{\mathrm{h}, \text{"yes"}, \text{"no"}\}) \times (\Sigma \times \{\rightarrow, \leftarrow, -\})^k$

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio

- $\delta(q, \sigma_1, \ldots, \sigma_k) = (p, \rho_1, D_1, \ldots, \rho_k, D_k)$
  If $M$ is in the state $q$, the cursor of the first string is scanning $\sigma_1$, that of the second $\sigma_2$ and so on, then the next state is $p$, the first cursor will write $\rho_1$ and move $D_1$ and so on.
- A configuration: $(q, w_1, u_1, \ldots, w_k, u_k)$
- $\overset{M}{\rightarrow}$ defined similarly as for the ordinary machine.
- A $k$-string machine starts from the configuration $(s, \triangleright, x, \triangleright, \epsilon, \ldots, \triangleright, \epsilon)$

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio



- Output is defined as for ordinary machines:
  If $(s, \triangleright, x, \triangleright, \epsilon, \ldots, \triangleright, \epsilon) \overset{M^*}{\rightarrow}$ ("yes", $w_1, u_1, \ldots, w_k, u_k$), then $M(x) =$ "yes".
  If $(s, \triangleright, x, \triangleright, \epsilon, \ldots, \triangleright, \epsilon) \overset{M^*}{\rightarrow}$ ("no", $w_1, u_1, \ldots, w_k, u_k$), then $M(x) =$ "no".
  If $(s, \triangleright, x, \triangleright, \epsilon, \ldots, \triangleright, \epsilon) \overset{M^*}{\rightarrow}$ $(\mathrm{h}, w_1, u_1, \ldots, w_k, u_k)$, then $M(x) = y$ where $y$ is $w_k u_k$ with the leading $\triangleright$ and trailing $\sqcup$s removed. (*Output* read from the *last ($k$th) string*.)
- The *time required* by $M$ on input $x$ is $t$ iff
  $(s, \triangleright, x, \triangleright, \epsilon, \ldots, \triangleright, \epsilon) \overset{M^t}{\rightarrow} (H, w_1, u_1, \ldots, w_k, u_k)$ where $H \in \{\mathrm{h}, \text{"yes"}, \text{"no"}\}$.
  If $M(x) = \nearrow$, then the time required is thought to be $\infty$.

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio



## Complexity classes

- Performance measured by the amount of time (or space) required on instances of size $n$ using a function of $n$.
- Machine $M$ *operates within time* $f(n)$ if for any input string $x$, the time required by $M$ on $x$ is at most $f(|x|)$.
- Function $f(n)$ is a *time bound* for $M$.
- A *complexity class* $\mathbf{TIME}(f(n))$ is a set of *languages* $L$ such that $L$ is decided by a multi string Turing machine operating within time $f(n)$.
- Notice that *worst-case inputs* are taken into account.

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio



## Multi string vs. single string machines

**Theorem.** Given any $k$-string Turing machine $M$ operating within time $f(n)$, we can construct a Turing machine $M'$ operating within time $\mathcal{O}(f(n)^2)$ and such that for any input $x$, $M(x) = M'(x)$.

Proof sketch:

- $M'$ simulates $M$ using one string by concatenating the strings of $M$: $(q, w_1, u_1, \ldots, w_k, u_k) \mapsto (q, \triangleright w'_1 u_1 \triangleleft w'_2 u_2 \triangleleft \ldots w'_k u_k \triangleleft \triangleleft)$
- Initially: $(s, \triangleright \underline{\triangleright'} x \triangleleft \underline{\triangleright'} \triangleleft \ldots \underline{\triangleright'} \triangleleft \triangleleft)$
- Simulation of a step of $M$ by $M'$:
  1. pass: symbols underlined (scanned) on the $k$ strings
  2. pass: change in the underlined (scanned) symbols

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio

- The strings of $M$ have a total length of $\mathcal{O}(kf(n))$.
  To simulate one step of $M$, $M'$ needs $\mathcal{O}(k^2 f(n))$ steps.
- $M$ makes at most $f(n)$ steps.
  $M'$ makes $\mathcal{O}(f(n)^2)$ steps (k is fixed and independent of $x$).

Thesis: No conceivable "realistic" improvement on the Turing machine will affect the speed more than polynomially.

© 2002 Teknillinen korkeakoulu, Tietojenkäsittelyteorian laboratorio