A Proof that INDEPENDENT SET is NP-complete

See Theorem 9.4 and its proof in the book.

A Proof that PARTITION INTO TRIANGLES is NP-complete

Preliminaries

Graphs in this exercise are assumed to be undirected and not containing self-loops.

The problem EXACT COVER BY 3-SETS is the following:

> We are given a family $F = \{S_1, \ldots, S_n\}$ of subsets of a set U, such that $|U| = 3m$ for some integer m, and $|S_i| = 3$ for all i. The question is whether there are m sets in F that are disjoint and have U as their union.

We know that EXACT COVER BY 3-SETS is **NP**-complete (see page 201 in Papadimitriou's book).

The reduction technique revisited: a way to show that a problem is **NP**-complete is to use the deduction rule below.

$$\frac{\begin{array}{c} L \text{ is } \mathbf{NP}\text{-complete} \\ L' \text{ is in } \mathbf{NP} \\ \text{There is a mapping } f \text{ such that a string } x \in L \text{ iff } f(x) \in L' \\ f(x) \text{ can be be computed in logarithmic space w.r.t. } |x| \end{array}}{L' \text{ is } \mathbf{NP}\text{-complete}}$$

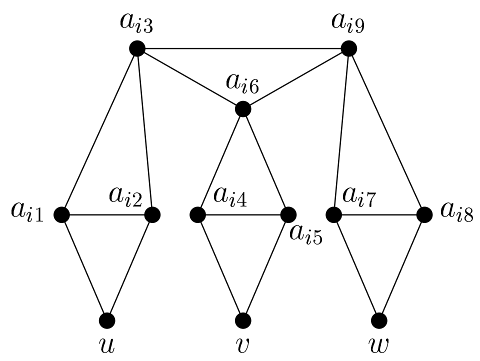
The Problem and a Solution

Show that the following problem, called PARTITION INTO TRIANGLES, is **NP**-complete:

> Given a graph $G = \langle V, E\rangle$ with $|V| = 3q$ for an integer q, is there a partition of V into q disjoint sets $V_1, V_2, \ldots, V_q$ of three vertices each such that, for each $V_i = \{v_{i1}, v_{i2}, v_{i3}\}$, the three edges $\{v_{i1}, v_{i2}\}$,$\{v_{i1}, v_{i3}\}$ and $\{v_{i2}, v_{i3}\}$ all belong to E?

Clearly PARTITION INTO TRIANGLES is in **NP** because we can first non-deterministically guess a disjoint partition of V and then check (in deterministic polynomial time) that each V_i fulfills the triangle condition.

We show the **NP**-hardness by reducing from the **NP**-complete problem EXACT COVER BY 3-SETS. Assume that we are given a set U such that $|U| = 3m$ for some integer m and a family $F = \{S_1, \ldots, S_n\}$ of subsets of U such that $|S_i| = 3$ for all i. We now construct a graph $G = \langle V, E\rangle$ with $|V| = 3q'$ such that G can be partitioned into q' triangles iff F contains an exact cover for U. We substitute for each set $S_i = \{u, v, w\}$ appearing in F the collection E_i of 18 edges shown below.



The graph $G = \langle V, E\rangle$ is therefore defined by

$$\begin{aligned} V &= U \cup \bigcup_{i=1}^{n} \{a_{ij} \mid 1 \le j \le 9\} \\ E &= \bigcup_{i=1}^{n} E_i. \end{aligned}$$

Now $|V| = |U| + 9n = 3m + 9n$ and thus $q' = m + 3n$. Note that only the U-vertices may be shared by different E_is.

Assume that $F' \subseteq F$ is an exact partition of U (and thus $|F'| = m$). Then, if $S_i = \{u, v, w\} \in F'$, take $\{a_{i3}, a_{i6}, a_{i9}\}$, $\{u, a_{i1}, a_{i2}\}$, $\{v, a_{i4}, a_{i5}\}$ and $\{w, a_{i7}, a_{i8}\}$ to belong to the triangle partition of G and, if $S_i = \{u, v, w\} \notin$

F', choose $\{a_{i1}, a_{i2}, a_{i3}\}$, $\{a_{i4}, a_{i5}, a_{i6}\}$ and $\{a_{i7}, a_{i8}, a_{i9}\}$ to belong to the triangle partition of G. Clearly the result is a valid triangle partition of G because the partition is disjoint, each vertex is in the partition and the partition has $q' = m + 3n$ sets.

To show the opposite direction, assume that V can be partitioned into $q' = m + 3n$ triangles $V_1, V_2, \ldots, V_q$. Then the corresponding exact cover of U is given by choosing those S_i of F such that $\{a_{i3}, a_{i6}, a_{i9}\} = V_k$ for some $1 \leq k \leq q$. This is justified as follows. For each $u \in U$ is there is exactly one triangle in which u belongs, let's say it was $\{u, a_{i1}, a_{i2}\}$. Then $\{a_{i3}, a_{i6}, a_{i9}\}$, $\{v, a_{i5}, a_{i6}\}$ and $\{w, a_{i6}, a_{i9}\}$ for $S_i = \{u, v, w\}$ must also be in the triangle partition. Furthermore, since no other $\{u, a_{jx}, a_{jy}\}$ can be in the triangle partition, the corresponding $\{a_{j3}, a_{j6}, a_{j9}\}$ cannot be in the triangle partition. Therefore each u appears in exactly one chosen S_i. This also implies that we choose exacly m S_is.

We have thus showed that G can be partitioned into $q' = m + 3n$ triangles iff F contains an exact cover of m sets for U.

To show that the reduction can be carried out in logarithmic space, note that the reduction is made by local substitutions. We therefore need 3 registers of length $\log(3m)$ to remember the elements in the currently processed set S_i of F and one counter of length $\log n$ to remember the number i of the set. We then just output the edges in E_i to the output tape.

A Note

A question was raised why we have to proof the correctness of a reduction in two directions, for instance that a family F of subsets of U contains an exact cover for U iff the constructed graph G can be partitioned into triangles. Now assume that we only showed the "if" direction. Then, if we solved the problem of whether G can be partitioned into triangles and the answer was "yes", we could not say anything about the truth of the original exact cover problem for F and U since it could be "yes" or "no" (however, if the answer was "no", then we could conclude that F does not contain an exact cover for U – therefore we obtain only partial information about the original problem). The similar case applies to the "only if" direction.