Slide 1

Boolean Logic

(Propositional logic)

- Syntax: Boolean variables (atoms), connectives
  Boolean expressions (formulae): $(x_1 \vee x_2) \wedge \neg x_3$
- Semantics: truth assignment $T : X \mapsto \{\mathbf{True}, \mathbf{False}\}$
  Let $T(x_1) = \mathbf{True}$, $T(x_2) = \mathbf{False}$.
  Then $T \models x_1 \vee x_2$, $T \not\models x_1 \wedge x_2$
- ϕ_1 and ϕ_2 equivalent ($\phi_1 \equiv \phi_2$)
  iff for all truth assignments T, $T \models \phi_1$ iff $T \models \phi_2$.

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- Boolean expression is equivalent to one in conjunctive (disjunctive) normal form.
  CNF: $(l_{11} \vee \cdots \vee l_{1n}) \wedge \cdots \wedge (l_{m1} \vee \cdots \vee l_{mn})$
  DNF: $(l_{11} \wedge \cdots \wedge l_{1n}) \vee \cdots \vee (l_{m1} \wedge \cdots \wedge l_{mn})$
  where each l_{ij} is a literal (Boolean variable or its negation).
- ϕ is satisfiable iff there is a truth assignment T such that $T \models \phi$.
- ϕ is valid iff for every truth assignment T, $T \models \phi$.
- ϕ is valid iff $\neg\phi$ is not satisfiable.

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- SAT problem:
  Given a Boolean expression ϕ in CNF, is it satisfiable?
  Example: $(x_1 \vee \neg x_2) \wedge \neg x_1$ is satisfiable
  $(x_1 \vee \neg x_2) \wedge \neg x_1 \wedge x_2$ is not.
- SAT can be solved in $n^2 2^n$ time straightforwardly
- SAT $\in \mathbf{NP}$
- HORNSAT problem:
  Given a conjunction of Horn clauses, is it satisfiable?
  Horn clause is a clause (disjunction of literals) with at most one positive literal: $\neg x_1 \vee x_2 \vee \neg x_3$, $\neg x_1 \vee \neg x_3$, x_2
- HORNSAT $\in \mathbf{P}$

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Boolean Circuits

Syntax

Graph $C = (V, E)$, $V = \{1, 2, \ldots, n\}$ (gates)
C acyclic (for all $(i, j) \in E, i < j$).

All gates i have a sort
$s(i) \in \{\mathbf{True}, \mathbf{False}, \wedge, \vee, \neg\} \cup \{x_1, x_2, \ldots\}$.

if $s(i) \in \{\mathbf{True}, \mathbf{False}\} \cup \{x_1, x_2, \ldots\}$, indegree 0 (inputs)
if $s(i) = \neg$, indegree 1
if $s(i) \in \{\vee, \wedge\}$, indegree 2
Node n output

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Semantics

$X(C)$: variables in C

Truth assignment $T : X(C) \mapsto \{\mathbf{True}, \mathbf{False}\}$

Truth value $T(i)$ for each gate i inductively:
If $s(i) = \mathbf{True}$, then $T(i) = \mathbf{True}$.
If $s(i) = \mathbf{False}$, then $T(i) = \mathbf{False}$.
If $s(i) \in X(C)$, then $T(i) = T(s(i))$.
If $s(i) = \wedge$, then $T(i) = \mathbf{True}$ if $T(j) = T(j') = \mathbf{True}$, otherwise $T(i) = \mathbf{False}$ where (j, i) and (j', i) are the edges entering i.

$\vdots$

$T(C) = T(n)$

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- CIRCUIT SAT:
  Given a circuit C, is there a truth assignment T such that $T(C) = \mathbf{True}$?
  CIRCUIT SAT $\in$ $\mathbf{NP}$
- CIRCUIT VALUE:
  Given a circuit C with no variables, is there a truth assignment T such that $T(C) = \mathbf{True}$?
  CIRCUIT VALUE $\in$ $\mathbf{P}$
  (No truth assignment is needed).