T-79.231 Parallel and Distributed Digital Systems

Structural Analysis

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Structural Analysis vs. Reachability Analysis

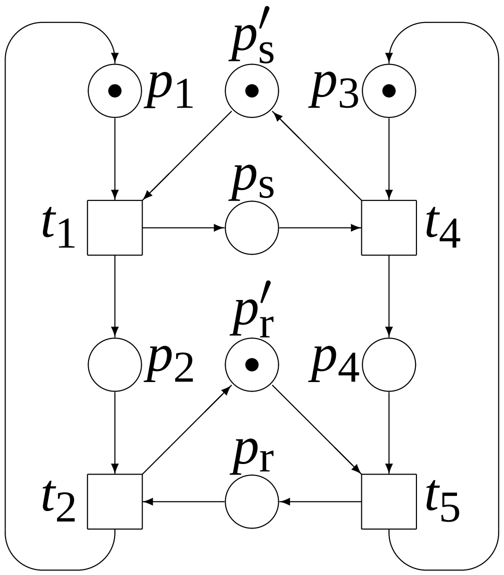
It is fairly easy to implement reachability analysis for any system that performs computations. In practice, its applicability is limited by the large number of reachable states. Reachability analysis starts from a predetermined initial state.

Structural analysis makes it possible to prove certain properties based on the structure of the model. The analysis results may hold for several different initial states or instances of the model. If a compiler of a programming language warns about an unused variable or procedure, they are redundant for every input of the program.

This lecture concentrates on place/transition nets, whose structural analysis is based on a matrix representation of the net structure.

The Incidence Matrix

A place/transition net $\langle S,T,F,W\rangle$ can be transformed into a corresponding *insidence matrix* $A : T \times S \to \mathbb{Z}$. A row $t \in T$ of the matrix denotes how the firing of the transition t affects the marking of the net: $A(s,t) = W(\langle t,s\rangle) - W(\langle s,t\rangle)$.



$$A = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cc} \begin{array}{cccccccc} p_1 & p_2 & p_3 & p_4 & p_s & p'_s & p_r & p'_r \end{array} & \\ \begin{pmatrix} -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \end{pmatrix} & \begin{array}{c} t_1 \\ t_2 \\ t_4 \\ t_5 \end{array} \end{array}$$

Here is a net and its incidence matrix. As there is at most one arc between a given place and transition, the arc weights do not cancel out, so they are visible as the elements of the matrix.

The Firing Rule with Matrices

Let us now see what happens when each of the transitions t_1, t_4 and t_5 fires once in our example system, starting from the initial marking $M_0 = (1\ 0\ 1\ 0\ 0\ 1\ 0\ 1)^{\mathrm{T}}$. Let us denote the firing times with the row vector $u = (1\ 0\ 1\ 1)$ and let us compute $M = M_0 + A^{\mathrm{T}} u$.

$$M = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

The Incidence Matrix and Bi-Directional Arcs

In the incidence matrix, the weights of bi-directional arcs between a place and a transition cancel out. If our example net is augmented with arcs from t_1 to p_4 and from p_4 to t_1, the initial marking becomes a deadlock. However, the addition of these two arcs has no effect on the incidence matrix.

If the net contains bi-directional arcs, its real behaviour (reachability graph) may be a subset of the behaviour that can be inferred from the incidence matrix.

Place Invariants (S-Invariants) (1/3)

The solutions y of the set of linear equations $Ay = 0$ are *place invariants*. Example:

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

It is easy to see that $y_2 = y_1$, $y_4 = y_3$, $y_6 = y_5$ and $y_8 = y_7$ solve the system. For instance, the matrix equation holds for $y = (y_1\ y_1\ 0\ 0\ 0\ 0\ 0\ 0)^{\mathrm{T}}$. Let us remove the solved columns:

$$\begin{aligned} -y_1 + y_5 &= 0 \qquad & -y_3 - y_5 &= 0 \\ y_1 - y_7 &= 0 \qquad & y_3 + y_7 &= 0 \end{aligned}$$

One more solution is obtained: $y_5 = y_7 = y_1 = -y_3$, or $y = (y_1\ 0\ -y_1\ 0\ y_1\ 0\ y_1\ 0)^{\mathrm{T}}$.

Place Invariants (S-Invariants) (2/3)

Each solution to our example contains an unconstrained multiplier. It is customary to omit such multipliers, as the linear combinations of invariants are invariants. If y is a place invariant, it holds in every marking M reachable from the initial marking M_0 that $M^{\mathrm{T}}y = M_0^{\mathrm{T}}y$. Let us display the invariants in a more readable form $M^{\mathrm{T}}y$:

$$M(p_1) + M(p_2) \tag{1}$$

$$M(p_3) + M(p_4) \tag{2}$$

$$M(p_{\mathrm{s}}) + M(p_{\mathrm{p}'}) \tag{3}$$

$$M(p_{\mathrm{r}}) + M(p_{\mathrm{v}'}) \tag{4}$$

$$M(p_1) - M(p_3) + M(p_{\mathrm{s}}) + M(p_{\mathrm{r}}) \tag{5}$$

The negative term of the invariant (5) can be eliminated by adding (2): $M(p_1) + M(p_4) + M(p_{\mathrm{s}}) + M(p_{\mathrm{r}})$. In the initial marking of the picture, this expression evaluates to 1. Thus, exactly one of the places p_1, p_4, p_{s} or p_{r} is marked.

Place Invariants (S-Invariants) (3/3)

Place invariants denote the relations between places as weighed sum expressions. They can be used to prove certain safety properties, such as the amount of tokens is bounded. Place invariants can also be utilised in more complex proofs.

Even in reachability analysis, place invariants can be helpful. If the marking of a place is normally represented with n bits, the markings of our example net need $8n$ bits of storage. However, according to the first four place invariants, the places paikat p_2, p_4, $p_{\mathrm{p}'}$ and $p_{\mathrm{v}'}$ are complement places of p_1, p_3, p_s and p_r, and thus their markings need not be stored. The markings of the remaining places would consume $4n$ bits, unless we observed that $M(p_1)+M(p_4)+M(p_\mathrm{s})+M(p_\mathrm{r}) = 1$. This observation allows us to represent the marking of the net with 2 bits.

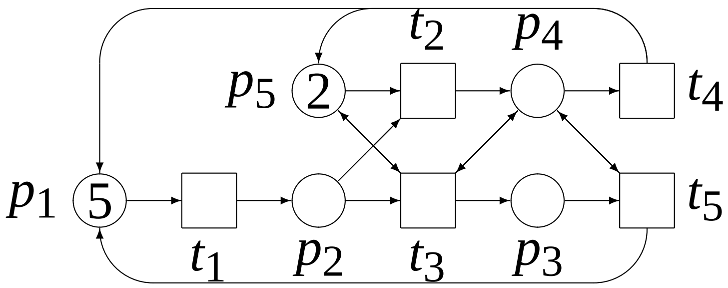
Transition Invariants (T-Invariants) (1/2)

Transition invariants x solve the equation $A^{\mathrm{T}}x = 0$. They indicate loops. Example:

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

It looks like $x_1 = x_2$, $x_1 = x_3$, $x_3 = x_4$ and $x_2 = x_4$. None of these alone solves the matrix equation: for instance, $A^{\mathrm{T}}(1\ 1\ 0\ 0)^{\mathrm{T}} = (0\ 0\ 0\ 0\ 1\ -1\ -1\ 1)$. Thus, the only solution is $x = (x_1\ x_1\ x_1\ x_1)$. In other words, the markings of this net recur if each transition fires the same number of times.

Transition Invariants (T-Invariants) (2/2)



$$A = \begin{array}{c} \\ \left(\begin{array}{ccccc} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & 0 \end{array}\right) \end{array}
\begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array}$$

with columns labelled $p_1 \; p_2 \; p_3 \; p_4 \; p_5$.

The incidence matrix corresponding to the model of a multi-processor system does not represent all firing conditions of the transitions t_3 and t_5, as there are bi-directional arcs attached to them. The solutions of the place invariant equation $Ay = 0$ are $y_1 = y_2 = y_3 = y_4 - y_5$—in other words, $y = (1\ 1\ 1\ 1\ 0)^{\mathrm{T}}$ or $y = (0\ 0\ 0\ 1\ 1)^{\mathrm{T}}$.

The solution of the transition invariant equation $A^{\mathrm{T}}x = 0$ is $x_1 = x_2 + x_3$, $x_4 = x_2$ and $x_5 = x_3$—that is, $x = (1\ 1\ 0\ 1\ 0)^{\mathrm{T}}$ or $x = (1\ 0\ 1\ 0\ 1)^{\mathrm{T}}$. The loops of the net are thus t_1, t_2, t_4 and t_1, t_3, t_5.

Also linear combinations of transition invariants are invariants. For instance, firing t_1 six times and other transitions thrice, that is, $x = (6\ 3\ 3\ 3\ 3)^{\mathrm{T}}$, has no effect on the marking.

Summary

With structural analysis, it is possible to prove properties of a model without exploring its dynamic behaviour.

The structural analysis of place/transition nets is based on incidence matrices $A(s,t) = W(\langle t,s\rangle) - W(\langle s,t\rangle)$ and the solutions of linear equation groups $Ay = 0$ or $A^{\mathrm{T}}x = 0$. Place invariants are solutions y and transition invariants are solutions x. For a given initial marking of the net, invariant expressions are constant in every reachable marking.

Invariants provide a mechanism for checking that the model has been constructed correctly. For instance, if the net is supposed to contain a loop, the vector x corresponding to it must fulfil the equation $A^{\mathrm{T}}x = 0$.