1. First, we construct the incidence matrix of A:

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix}$$

The columns represent the places in the order $i_1, w_1, a_1, i_2, w_2, a_2, p$. The rows represent the transitions in the order $s_1, s_2, b_1, b_2, e_1, e_2$.

The task is to show that the places a_1 and a_2 can never be marked at the same time. If the marking of places a_1, a_2 and p is invariant, that is, $M(a_1) + M(a_2) + M(p) = k$ in all markings, then $M(a_1) + M(a_2) \leq k$ in all markings. Thus, we must show that $A \cdot \iota = \mathbf{0}$. ι is the vector representing the places belonging to the place invariant.

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By calculating the product, we see that the result is indeed a zero vector. Now we can calculate k in the following way: $M_0^T \cdot \iota = k$. By performing the calculation we get

$$k = (1\ 0\ 0\ 1\ 0\ 0\ 1) \cdot (0\ 0\ 1\ 0\ 0\ 1\ 1)^T = 1$$

from which follows that $M(a_1) + M(a_2) \leq 1$. This means that in the places a_1 and a_2 there can be at most one token at any marking.

In general, the place invariants can be calculated by solving the matrix equation $A \cdot \mathbf{y} = \mathbf{0}$. From the solution one can construct the basis for place invariants, that is, vectors from which we can construct all possible place invariants of the net by linear combination.

The net of this assignment gives a solution

$$\begin{pmatrix} y_1 - y_3 \\ y_1 - y_3 \\ y_1 \\ y_2 - y_3 \\ y_2 - y_3 \\ y_2 \\ y_3 \end{pmatrix}$$

Now we can get the invariant for mutual exclusion by assigning $y_1 = y_2 = y_3$ when the vector will become $(0\ 0\ y_1\ 0\ 0\ y_1\ y_1)^T$. One can get the other solutions by assigning $y_2 = y_3 = 0$ or $y_1 = y_3 = 0$, when the vectors will be $(1\ 1\ 1\ 0\ 0\ 0\ 0)^T$ and $(0\ 0\ 0\ 1\ 1\ 1\ 0)^T$. These three vectors form the kernel of the incidence matrix A, and all invariants of the net can be constructed from them with linear combination.

2. The place invariant method in itself can not be used to show mutual exclusion because the algorithm uses loops. Loops are not visible in the incidence matrix, thus they can not be modelled with invariants. Removing all loops from the net would destroy the mutex property but the place invariants would remain. The solution can be found from the notion of *trap*, which is defined in the tutorial paper.

   The mutual exclusion can be represented as an inequality $M(w)+M(r) \leq 1$. To show that the inequality holds, we need two place invariants: $M(r)+M(r')=1$ and $M(w)+M(w')=1$. It is fairly easy to see that the equations could be invariants. We will check them anyway:

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix}$$

   The order of places is $i_w, wait_w, w, w', i_r, wait_r, r, r'$ and the order of transitions is s_w, bw, ew, s_r, br, er. Now $A \cdot (0\ 0\ 1\ 1\ 0\ 0\ 0\ 0)^T = \mathbf{0}$ and $A \cdot (0\ 0\ 0\ 0\ 0\ 0\ 1\ 1)^T = \mathbf{0}$. Therefore the equations $M(r)+M(r')$ and $M(w)+M(w')$ are invariant. Also

$$(1\,0\,0\,1\,1\,0\,0\,1) \cdot (0\,0\,1\,1\,0\,0\,0\,0)^T = 1$$

   and

$$(1\,0\,0\,1\,1\,0\,0\,1) \cdot (0\,0\,0\,0\,0\,0\,1\,1)^T = 1$$

   from which follows $M(l)+M(l')=1$ and $M(k)+M(k')=1$.

   In addition to invariants we need also the trap inequality. We guess that $P=\{r', w'\}$ could be a trap. Check: $P^\bullet = \{bw, br\}$ and ${}^\bullet P = \{bw, ew, br, er\}$. Now, $P^\bullet \subseteq {}^\bullet P$, therefore P is a trap and an initialized one, too, because $M_0(r') = M_0(w') = 1$. Thus, we can use the trap inequality
   $M(r') + M(w') \geq 1$.

   Now we have two equations and one inequality:

$$\begin{aligned} M(r') + M(w') &\geq 1 \\ M(r) + M(r') &= 1 \\ M(w) + M(w') &= 1 \end{aligned}$$

   By adding the equations and subtracting the inequality we get

$$M(r) + M(r') + M(w) + M(w') - M(r') - M(w') \leq 1 + 1 - 1$$

   which will become

$$M(r) + M(w) \leq 1$$

   which proves the mutual exclusion.