T-79.231 Parallel and Distributed Digital Systems

Stochastic Analysis

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Why is Stochastic Analysis Used?

- Ordinary Petri net reachability analysis and associated model checking methods can verify a system's correct behaviour

- Today, the correct functioning of a system is not enough. A system must exhibit reasonable performance in addition to correct functioning.

- The theory of stochastic Petri nets which dates back to mid-1980s has proven successful in the performance oriented modelling and analysis of small and medium sized systems

Analysis Tools for Stochastic Petri Nets

- Three analysers for Stochastic Petri Nets are listed in the following. These tools are free for academic use.
  - DSPNExpress
  - GreatSPN
  - TimeNET

Stochastic Petri Nets versus Stochastic Processes (1/2)

- Stochastic Petri nets are structurally identical to place transition nets
- The difference between stochastic Petri nets and place transition nets is that transitions may not fire arbitrarily, but that the firing delay of an enabled transition follows a given distribution. (Another difference is the inclusion of inhibitor arcs in stochastic nets. This point is not elaborated on here).
- The delay between a transition's becoming enabled and its firing follows the Exponential distribution. Because this distribution is *memoryless*, the conflicts between enabled transitions are easy to handle.

Stochastic Petri Nets versus Stochastic Processes (2/2)

- A net, where all transitions' firing delays follow the exponential distribution models a continuous time Markov chain. These nets are called (ordinary) stochastic Petri Nets, sometimes abbreviated as SPN.

- A net, where the transitions' firing delays either follow the exponential distribution or are immediate models a more general stochastic process. In mathematics we talk about semi-Markov processes and renewal processes. These kinds of nets are called Generalised Stochastic Petri Nets (GSPN).

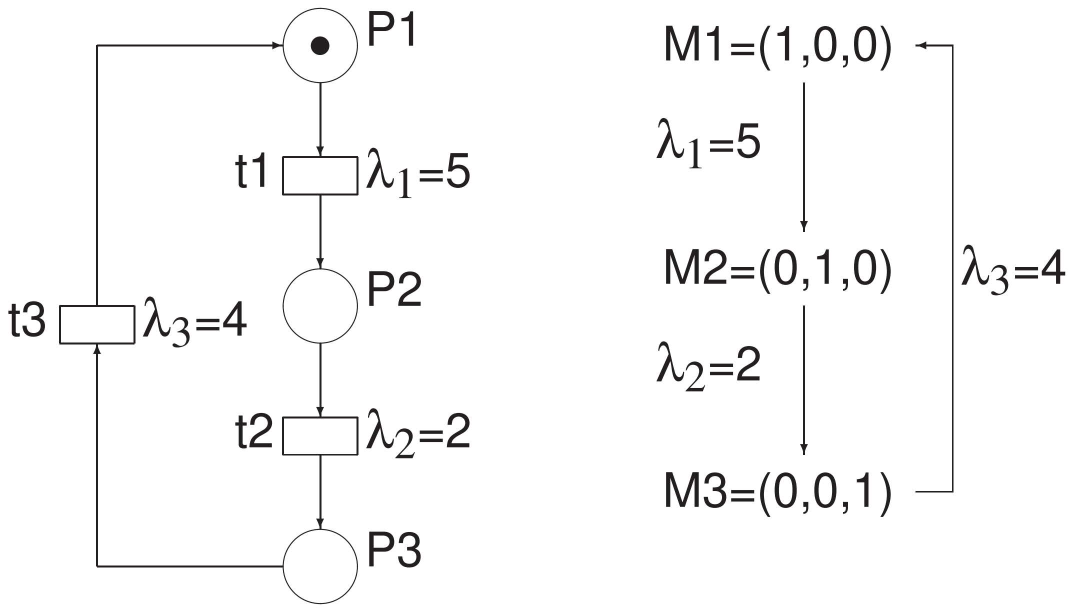
How are Stochastic Petri Nets analysed?

Stochastic Petri nets (here, SPNs without inhibitor arcs) are analysed in a three phase fashion. These are:

1. We generate the net's reachability graph, treating it as a regular place transition net.
2. In the resulting reachability graph we mark each transition's firing frequency (the only parameter needed for specifying the exponential distribution). Using this info, we can form the state transition matrix.
3. Using standard linear algebraic techniques, we solve the steady state of the Q-matrix, and use it to compute the expected numbers of tokens in certain places, etc.

In the following, we show by an example how stochastic analysis works.

Example Net with its Reachability Graph



Example Continues

From the previous slide's reachability graph we can form the Q matrix of the Markov process. The result is

$$Q = \begin{pmatrix} -5 & 5 & 0 \\ 0 & -2 & 2 \\ 4 & 0 & -4 \end{pmatrix}$$

As a note, the elements on the diagonal are obtained from the formula:

$$q_{ii} = -\sum_{j \neq i} q_{ij}$$

In other words, the element on the diagonal is chosen in such a way that the sum of each row is 0.

Following this, we compute the steady state of the Q matrix.

Computation of the Steady State (1/3)

Steady state π fulfils the following condition:

$$\begin{cases} \pi Q & = & 0^T \\ \sum_i \pi_i & = & 1 \end{cases}$$

Now, we will solve this group of equations for the example system.

$$\begin{cases} \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \begin{pmatrix} -5 & 5 & 0 \\ 0 & -2 & 2 \\ 4 & 0 & -4 \end{pmatrix} & = & \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \\ \pi_1 + \pi_2 + \pi_3 & = & 1 \end{cases}$$

Computation of the Steady State (2/3)

We get the following group of equations:

$$\begin{cases} -5\pi_1 + 4\pi_3 & = & 0 \\ 5\pi_1 - 2\pi_2 & = & 0 \\ 2\pi_2 - 4\pi_3 & = & 0 \\ \pi_1 + \pi_2 + \pi_3 & = & 1 \end{cases}$$

Let us consider the last two equations:

$$\begin{cases} 2\pi_2 - 4\pi_3 & = & 0 \\ \pi_3 & = & 1 - \pi_1 - \pi_2 \end{cases}$$

Computation of the Steady State (3/3)

By straight substitution, the last two equations yield

$$4 = 6\pi_2 + 4\pi_1$$

Now we can solve the linear group where the above equation is combined with the second equation from the original linear group.

$$\begin{cases} 4\pi_1 + 6\pi_2 & = & 4 \\ 5\pi_1 - 2\pi_2 & = & 0 \end{cases}$$

From this we get $\pi_1 = \frac{4}{19}$ and $\pi_2 = \frac{10}{19}$. By substituting into the third equation in the original group, we get $\pi_3 = \frac{5}{19}$.

How is the Steady State Used? (1/2)

From the previous computations we have derived the steady state of the example net as

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} = \begin{pmatrix} \frac{4}{19} & \frac{10}{19} & \frac{5}{19} \end{pmatrix}$$

In general the state probabilities may be used to compute the expected value of tokens in a particular place, how often a given transition fires, etc.

The formula for the expected value of tokens in a place is

$$E(\text{"tokens in place } s\text{"}) = \sum_i (\pi_i m_i(s)).$$

In the formula, m_i is the net marking, and $m_i(s)$ gives the number of tokens in place s in the marking m_i.

How is Steady State Used? (2/2)

The Steady State distribution can be utilised to determine how often a transition fires per unit of time during the execution of the net.

A formula for this rate of firing is:

$$f_j = \sum_{m_i \in B_j} \frac{\pi_i \lambda(t_j)}{-q_{ii}}$$

In this formula B_j is the set of markings where the transition j is enabled.

Stochastic Analysis in General

- The performance analysis of a system is a much harder problem than ascertaining that a system performs correctly.
- Stochastic analysis starts where reachability analysis ends.
- In the general case we must find the inverse of a $n \times n$ matrix to determine the steady state distribution of a system with n states in the reachability graph. This is certainly a bottleneck in the analysis.
- If inverting the matrix is computationally infeasible, stochastic analysis has to be carried out by analysers capable of simulation.