T-79.231 Parallel and Distributed Digital Systems

Structural Analysis

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Why Use Structural Analysis?

- Reachability problem for Petri Nets is computationally expensive

- Often we would like to prove properties of the net without performing the whole reachability analysis

- Structural analysis provides us with tools by which we can check non-trivial properties of the net without performing reachability analysis. As an example, we can check whether the sum of tokens in all the places of the Petri net is the same in every marking.

- In this exposition, we concentrate on place invariants.

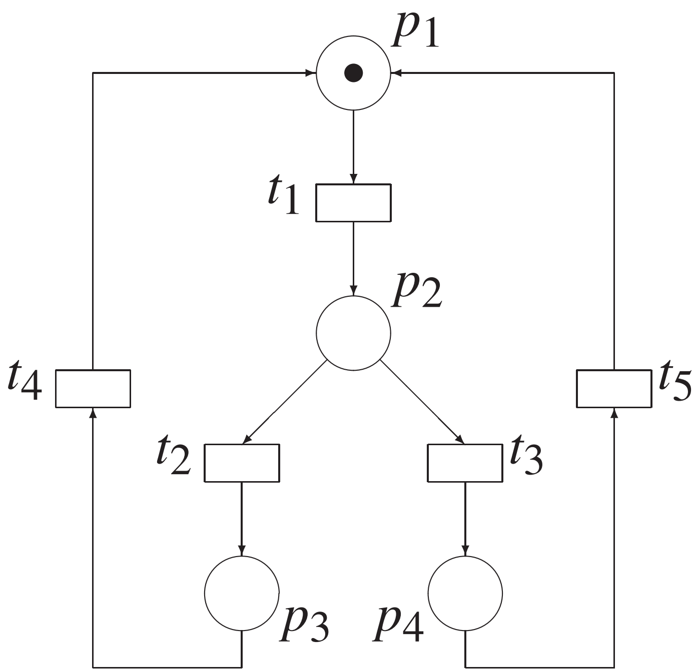
Prerequisites for Structural Analysis

- To perform structural analysis, We must be able to form the incidence matrix for each Place Transition net.
- Incidence matrix C is defined as follows:

  $$C : S \times T \longrightarrow \mathbb{Z} : C(\langle s,t \rangle) = W(\langle t,s \rangle) - W(\langle s,t \rangle)$$

  In the preceding formula, $W : (S \times T) \cup (T \times S) \rightarrow \mathbb{N}$ defines the arc weights in the net. If an arc is missing, the weight is 0.
- Incidence matrix is a concise representation for arc weights. However, it is not able to represent bidirectional arcs between places and transitions.
- Once the Incidence Matrix has been formed, all structural analyses can be carried out using standard linear algebraic methods.

Example of Place Invariants



Notice that the sum of tokens is bounded in every marking of the above net. Thus, the reachability graph of the net is finite.

Place Invariant Example Continues...

To prove that the sum of tokens in every marking of the net is constant (and equal to 1), we first determine the net's incidence matrix and perform some linear algebra.

We obtain the following incidence matrix:

$$C = \begin{pmatrix} -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

Place Invariant Example Continues. . .

Now we want to prove that $M(p_1)+M(p_2)+M(p_3)+M(p_4)$ is constant. To this end we define a column vector f, which is simply

$$f = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Now to prove that the expression is really an invariant, we have to verify that

$$f^T C = 0^T$$

That is, the transpose of f times the incidence matrix gives us the 0 vector.

Place Invariant Example Continues…

Using matrix operations we can easily determine that

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

Place Invariant Example Continues...

The form of the column vector f depends entirely on the invariant we want to prove. If we had wanted to prove that $M(p_1) + 3M(p_2)$ is an invariant, the f column vector would have taken the following form:

$$f = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}$$

If we do the computations, we will notice that the above expression is NOT an invariant

Transition Invariants

- Occasionally we are interested in knowing which transitions one has to fire in order for the net to return to the marking we left from.

- Here, the transition invariants (T-invariants) enter into the picture.

- T-invariants satisfy the following equation:

$$Cf = 0$$

where C is the aforementioned incidence matrix, and f is a column vector whose component i tells how many times transition i has been fired.

T-invariants Continue. . .

In the example net we notice that

$$f = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

is a T-invariant, because it satisfies the equation on the last slide. The T-invariant in question could be interpreted as follows: When we fire t_1 once, t_2 once and t_4 once, we end up in the same marking as where we left from.

T-invariants can be used to find cycles in the net's behaviour.

Conclusion

- Structural analysis is a quick way of checking important properties of nets.
- With place invariants we can verify that some linear combinations of token counts in places give the same result in every reachable marking
- With transition invariants we may compute cycles in the net's behaviour without doing reachability analysis.
- Certain net reduction techniques are based on invariants.