1 PERFORMANCE ANALYSIS

- In performance analysis, two approaches are feasible:
  1. Measurement.
     - Benchmarking.
     - Measurement.
     - Prototyping.
  2. Modelling.
     - Simulation models.
     - Analytical models.
- In the following treatise we consider how Petri nets can be used to construct analytical models.

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Introduction

- A model is either
  - Deterministic, or
  - Probabilistic.
- The probabilistic approach is often valid because:
  - Some details are not known,
  - Too much detail leads to too large a model,
  - We are satisfied with 'sufficiently' accurate results, and
  - The system's sensitivity to changes in the parameters can be investigated.

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Markov processes

- Probabilistic models have *stochastic processes* as their foundation.
- A stochastic process $\{X(t), t \in T\}$ is a set of random variable indexed with the parameter t.
- The random variable X can be modelled with its *probability density function* (pdf):

$$f_X(x) = \frac{d}{dx} P\{X \leq x\} \quad -\infty < x < \infty$$

- A stochastic process for which the following holds:

$$\begin{aligned} & P\{X(t) \leq x | X(t_n) = x_n, \ldots, X(t_0) = x_0\} \\ = \quad & P\{X(t) \leq x | X(t_n) = x_n\} \end{aligned}$$

  when $t > t_n > t_{n-1} > \ldots > t_0$ is a *Markov process*.
- In a Markov process:
  - The sojourn time in a certain state is exponentially distributed, and
  - Future behaviour is only dependent on the current state, not the history.

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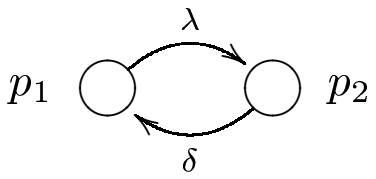
Markov processes

- A Markov process can be depicted as a state transition diagram:



  or a matrix:

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \delta & -\delta \end{pmatrix}$$

- the *steady state* probability distribution of an *ergodic* Markov process is obtained from the following system of equations:

$$\begin{aligned} \pi \cdot Q &= 0 \\ \Sigma_i \pi_i &= 1 \end{aligned}$$

  where π_i is the probability for being in the state i.

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Stochastic Petri nets

- Markov processes are problematic as a system model because they treat the state space directly.
- If a transition of a P/T net is associated with an exponential probability distribution to model the firing time of the transition, the reachability graph can be interpreted as a Markov process.

Definition 1 A stochastic Petri net (SPN) is the tuple $SPN =< S, T; F, K, M_0, W, \Lambda >$, where $< S, T; F, K, M_0, W >$ is the underlying net and $\Lambda : T \to \mathbb{R}$ expresses the *firing rate* of the transitions.

- Stochastic Petri nets are analysed by working on the reachability graph of the underlying net.
- A stochastic Petri net is *reversible* iff

$$M_0 \in M_i[> \quad \text{for all} \quad M_i \in M_0[>$$

- A reversible SP creates the state transition matrix Q of an ergodic Markov process in the following fashion:
  $q_{ij} = \Sigma_{t \in B} \Lambda(t)$, where
  $B = \{t \mid t \in T \wedge M_i[t > M_j \wedge i \neq j\}$, otherwise
  $q_{ij} = 0$ and q_{ii} are defined such that $\Sigma_j q_{ij} = 0$.

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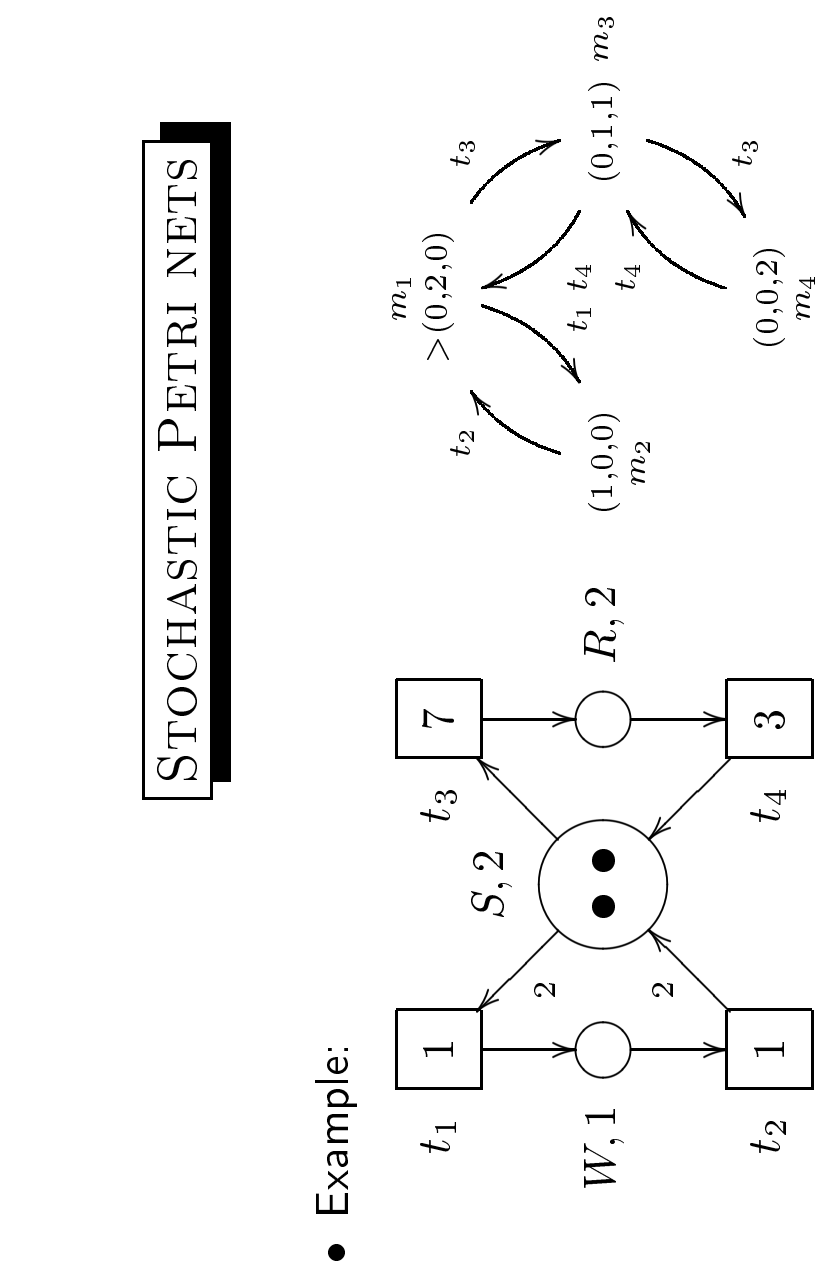
Stochastic Petri nets

- Example:



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Stochastic Petri nets

- The steady state matrix is given below:

$$Q = \begin{pmatrix} -(1+7) & 1 & 7 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 0 & -(3+7) & 7 \\ 0 & 0 & 3 & -3 \end{pmatrix} = \begin{pmatrix} -8 & 1 & 7 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 0 & -10 & 7 \\ 0 & 0 & 3 & -3 \end{pmatrix}$$

- The system of equations

$$\begin{aligned} \pi \cdot Q &= \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix} \cdot Q = 0 \\ \Sigma_i \pi_i &= 1 \end{aligned}$$

gives the following solutions: $\pi_1 = \pi_2 = \frac{9}{88} \approx 0,1$, $\pi_3 = \frac{21}{88} \approx 0,239$, and $\pi_4 = \frac{49}{88} \approx 0,557$.

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Stochastic Petri nets

- Now we will be able to compute the following probabilistic characterizations of the system:
  1. The probability of a condition:

     $$PB = \Sigma_{i \in B} \pi_i$$

     E.g., the condition "$|m(s)| \leq 1$" holds in the markings m_2, m_3. The probability for the condition to hold is: $\pi_2 + \pi_3 \approx 0,339$.
  2. The average value of tokens in a place:

     $$E[p_i] = \Sigma_{n=1}^{k}[n \cdot P\{B(i,n)\}]$$

     where $B(i,n)$ is a subset of $M_0[>$ where in a k-bounded place p_i there are n tokens. E.g., $E[R] = \Sigma_{n=1}^{2}[n \cdot P\{B(R,n)\}] = \pi_3 + 2\pi_4 \approx 1,353$.
  3. The average firing count of a transition:

     $$f_j = \Sigma_{m_i \in B_j} \pi_i (\frac{\lambda(t_j)}{-q_{ii}})$$

     where B_j is the set of those markings where t_j is in concession and q_{ii} is the sum of firing times of the transisitions which are in concession in the marking m_i. Example: $f_3 = \Sigma_{m_i \in B_3} \pi_i (\frac{\lambda(t_3)}{-q_{ii}}) = \pi_1 \cdot \frac{7}{1+7} + \pi_3 \cdot \frac{7}{7+3} \approx 0.1 \cdot 0.875 + 0.239 \cdot 0.7 \approx 0.255$ (Corrected $frac78$ to $\frac{7}{7+3}$!

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