1 STRUCTURAL ANALYSIS

- A structural property is a direct consequence of the topological structure of a Petri net.
- Observing a structural property does not require reachability analysis.
- Typically, analysis methods are based on linear algebra.
- In the following, we will consider certain safety properties which may be too complicated for temporal logics based analysis.
- Another goal for structural analysis is to characterise subclasses of Petri nets for which reachability analysis is easier.

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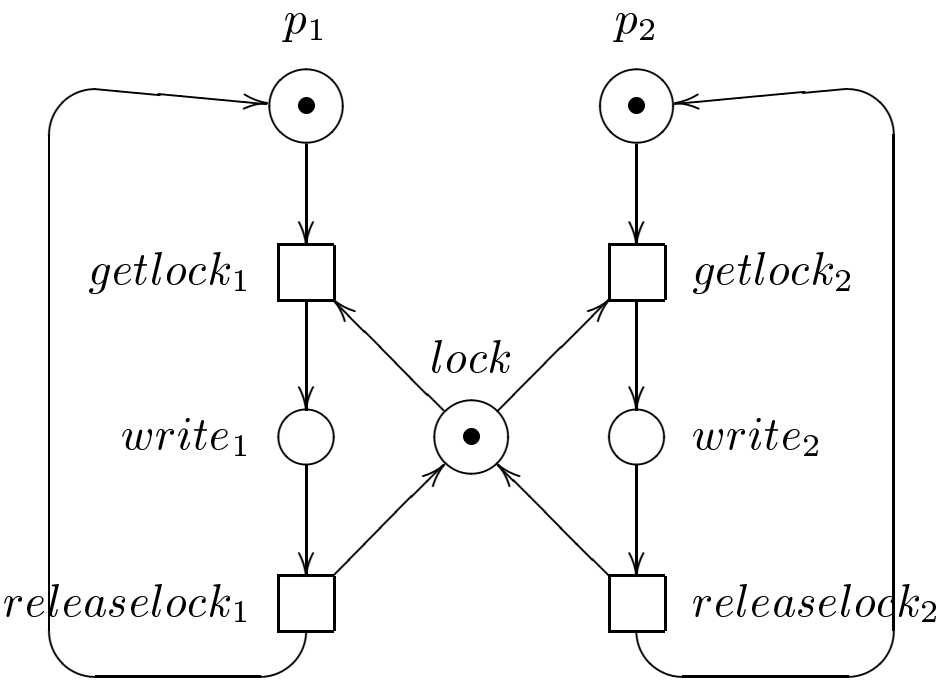
Introduction

- Consider the simple net below:



- We want to prove that the net never reaches a marking where both the place $write_1$ and $write_2$ have tokens.
- One of the solutions is to perform reachability analysis.
- On the other hand, it is easy to see from the structure of the net that the formula $(*)$
  $M(write_1) + M(write_2) + M(lock) = 1$ is valid in all markings.

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State equation

- The property presented by the formula $(*)$ is called a *place invariant*.
- Computation of invariants is based on the use of a *state equation*.

Definition 1 The *Parikh mapping* of the sequence $\sigma \in \Sigma^*$ is the mapping $\hat{\sigma} : \Sigma \to \mathbb{N}$, where $\hat{\sigma}(a)$ indicates the number of occurrences of $a \in \Sigma$ in the sequence σ.

Definition 2 The *incidence matrix* of a P/T net is the mapping

$$C : S \times T \to \mathbb{Z} :< s, t > \mapsto W(< t, s >) - W(< s, t >)$$

Theorem 3 Let N be a P/T net, C its incidence matrix, and σ one of the *transition sequences* which can be fired from the marking M and ending in the marking M'. The following holds:

$$M' = M + C \cdot \hat{\sigma}$$

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Invariants

Definition 4 The place vector ι is called a *place invariant* (S-invariant, P-invariant), iff

$$C^T \cdot \iota = 0.$$

The transition vector $\tau \geq 0$ is called a *transition invariant* (T-invariant), iff

$$C \cdot \tau = 0.$$

- When the state equation is multiplied with the place invariant we will obtain the followin:

  $$\iota^T \cdot M' = \iota^T \cdot M + \iota^T \cdot C \cdot \hat{\sigma} = \iota^T \cdot M + (C^T \cdot \iota)^T \cdot \hat{\sigma} = \iota^T \cdot M$$

  i.e. in all markings the product $\iota^T \cdot M$ is constant.

- Correspondingly, the following holds for the transition invariant and state equation:

  $$M' = M + C \cdot \tau = M$$

  i.e. if the transitions of the net can be fired the number of times expressed in the vector τ, we will return to the marking from which the firing started.

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Invariants

- Incidence matrix of our example:

$$C = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

- The invariant $(*)$ is expressed by the following vector:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \end{pmatrix}^T$$

where

$$C^T \cdot \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \end{pmatrix}^T = 0$$

- Mutual exclusion can be verified by the formula

$$i^T \cdot M_0 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 1$$

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Invariants

- By solving the equation

$$C^T \cdot \iota = 0.$$

  we will obtain all the invariants in the net.
- One of our problems is to find an interpretation to all the invariants.
- Invariants are more often used by finding an invariant property we want to check and then checking if the property really is an invariant.
- With invariants it is possible to express safety properties reflecting the capacities of places.

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