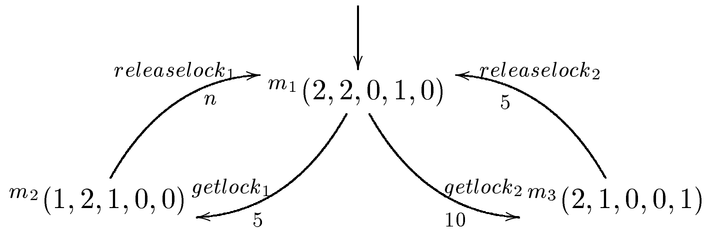
Tik-79.231 **Autumn 2000**
Parallel and distributed digital systems
Answers to Tutorial 6
Nov 10, 2000

1. First we will construct the reachability graph of the net:



Then we will create the infinitesimal generator matrix Q. (Row 1 models transition frequency from the state m_1 to the states m_2 and m_3. In the diagonal we have the negation of the sum of the rest of the elements in the row.)

$$Q = \begin{pmatrix} -15 & 5 & 10 \\ n & -n & 0 \\ 5 & 0 & -5 \end{pmatrix}$$

In the steady state the following conditions will hold: $\pi \cdot Q = 0 \wedge \sum_i \pi_i = 1$.

The steady state equations are obtained:

$$\begin{array}{rrrl} -15\pi_1 & +n\pi_2 & +5\pi_3 & = 0 \\ 5\pi_1 & -n\pi_2 & & = 0 \\ 10\pi_1 & & -5\pi_3 & = 0 \\ \pi_1 & +\pi_2 & +\pi_3 & = 1 \end{array}$$

The solution for the above system of equations is given below:

$$\pi_1 = \frac{n}{3n+5} \wedge \pi_2 = \frac{5}{3n+5} \wedge \pi_3 = \frac{2n}{3n+5}$$

With $n = 20$, we obtain the following:

$$\pi_1 = \frac{4}{13} \wedge \pi_2 = \frac{1}{13} \wedge \pi_3 = \frac{8}{13}$$

The average of tokens in the place p is obtained by addition of marking probabilities weighted with the number of tokens in the place p.

$$E[p_1] = \pi_1 \cdot m_1(p_1) + \pi_2 \cdot m_2(p_1) + \pi_3 \cdot m_3(p_1) = \frac{6n+5}{3n+5}$$

$$E[p_2] = \pi_1 \cdot m_1(p_2) + \pi_2 \cdot m_2(p_2) + \pi_3 \cdot m_3(p_2) = \frac{4n+10}{3n+5}$$

$$n = 20 \Rightarrow E[p_1] = 1\frac{12}{13} \wedge E[p_2] = 1\frac{5}{13}$$

$$n \to \infty \Rightarrow E[p_1] = 2 \wedge E[p_2] = \frac{4}{3}$$

$$n \to 0 \Rightarrow E[p_1] = 1 \wedge E[p_2] = 2$$

2. Place invariants fail to prove mutual exclusion because the net algorithm relies on loops. As loops do not show up in the incidence matrix, they cannot be represented and exploited by place invariants. Removing all the loops would destroy the mutex property but retain the place invariants. However, augmenting the place invariant technique with the notion of *traps* will provide a solution.

   Definition of trap. *Let $N = (S, T; F)$ be a P/T net and let $P \subseteq S$.*

   (a) P is a trap iff $P \neq \emptyset$ and $P^\bullet \subseteq {}^\bullet P$.

   (b) P is initialized iff $P \cap M_0 \neq \emptyset$.

   Definition of trap inequality. *Let $N = (S, T; F)$ be a P/T net and let $P = \{p_1, ..., p_k\}$ be a trap of N. Then*

   $$p_1 + ... + p_k \geq 1$$

   is the inequality *of P. The inequality of P is valid if the trap is initialized.*

   In other words, a trap is a new kind of structural property of Petri nets, providing new insight to the dynamic properties of the token game.

Mutex of the state testing algorithm

Mutual exclusion can be expressed as the inequality

$$writing + reading \leq 1.$$

Proof of the above inequation is based on the two place invariants (check correctness!) and trap inequality given below:

$$writing + canwrite = 1,$$

$$canread + reading = 1, \texttt{and}$$

$$canwrite + canread \geq 1.$$

Subtracting the trap inequality from the place invariant equations yields

$$writing + canwrite + canread + reading - canwrite - canread \leq 1 + 1 - 1$$

which immediately reduces to what we were supposed to prove.