T-79.230 **Kevät 2005**
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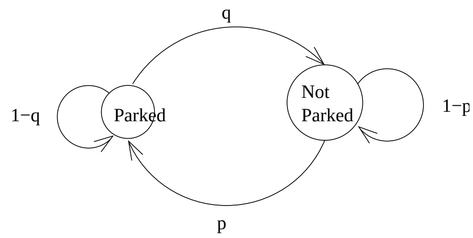
Solutions

1. A Fire station has one fire truck. When an emergency call comes, the truck goes out, fights fire and then returns.



We use shorthands p and q for the transition probabilities of the HMM illustrated in the figure.

(a) First, we need to figure out values for p and q.
For each one-hour time slice, the probability that the truck returns is p and the probability that is does not return is $1-p$. Thus the expected time that the truck is away is

$$\sum_{i=0}^{\infty}(i+1)\cdot p\cdot(1-p)^i = [\sum_{i=0}^{\infty}(1-p)^i]^2 = \frac{p}{(1-(1-p))^2} = \frac{1}{p}$$

hours. By equating this with three hours, we obtain $p=\frac{1}{3}$.
On the other hand, we assign $q=\frac{1}{12}$, as there is an emegency call once in twelwe hours on the average.

(b) $\mathbf{P}(Parked_{t+1} \mid Parked_t) = \langle \frac{11}{12}, \frac{1}{12}\rangle$ and
$\mathbf{P}(Parked_{t+1} \mid \neg Parked_t) = \langle \frac{1}{3}, \frac{2}{3}\rangle$.

(c) Suppose that $\mathbf{P}(Parked_t) = \langle r, 1-r\rangle$ where r is introduced as a parameter probability. Using the transition model, we obtain

$$\begin{aligned}\mathbf{P}(Parked_{t+1}) &= r\cdot\langle 1-q, q\rangle + (1-r)\cdot\langle p, 1-p\rangle \\ &= \langle r\cdot(1-q)+(1-r)\cdot p, qr+(1-r)\cdot(1-p)\rangle \\ &= \langle r\cdot(1-p-q)+p, (1-r)\cdot(1-p-q)+q\rangle.\end{aligned}$$

In the long run, we have $\mathbf{P}(Parked_t) = \mathbf{P}(Parked_{t+1})$ from which we obtain $r = r\cdot(1-p-q)+p$. It follows that $r\cdot(1-1+p+q)=p$ and $r=\frac{p}{p+q}$. By substituing the known values for p and q, we obtain

$$r = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{12}} = \frac{4}{4+1} = \frac{4}{5}.$$

Thus the truck spends on the average $24\cdot\frac{4}{5}$ hours (i.e. 19 hours and 12 minutes) per day at the fire station in the long run.

(d) Our task is to derive an exact expression distribution $\mathbf{P}(Parked_t)$ when $\mathbf{P}(Parked_0) = \langle k,\, 1-k \rangle$. We may utilize the general transition probabilities from (c) and conclude the following probabilities:

$$\begin{array}{rcl} P(Parked_0) & = & k, \\ P(Parked_1) & = & k \cdot (1-p-q) + p, \\ P(Parked_2) & = & (k \cdot (1-p-q) + p) \cdot (1-p-q) + p \\ & = & k \cdot (1-p-q)^2 + p \cdot (1-p-q) + p, \\ & \cdots & \\ P(Parked_t) & = & k \cdot (1-p-q)^t + p \cdot \sum_{i=0}^{t-1} (1-p-q)^i. \end{array}$$

The case of $P(\neg Parked_t)$ can be handled by analogy and symmetry. By exchanging the roles of p and q as well as k and $k-1$, we obtain

$$P(\neg Parked_t) = (1-k) \cdot (1-p-q)^t + q \cdot \sum_{i=0}^{t-1} (1-p-q)^i.$$

Then we conclude that $\mathbf{P}(Parked_t) = \langle P(Parked_t),\, P(\neg Parked_t) \rangle$.

(e) First, it is easy to see from the expressions above that whenever $|1-p-q| < 1$ or (i.e. $0 < p+q < 2$) it holds that

$$\begin{array}{rcl} \lim_{t \to \infty} \mathbf{P}(Parked_t) & = & \langle p \cdot \frac{1}{1-(1-p-q)},\, q \cdot \frac{1}{1-(1-p-q)} \rangle \\ & = & \langle \frac{p}{p+q},\, \frac{q}{p+q} \rangle \\ & = & \langle r,\, 1-r \rangle \end{array}$$

for the probability r derived in (c). In particular, if $1-p-q = 0$, or equivalently $p+q = 1$, then we have $\lim_{t \to \infty} \mathbf{P}(Parked_t) = \langle p,\, q \rangle$. Second, if $1-p-q = 1$, then $p = q = 0$ and

$$\lim_{t \to \infty} \mathbf{P}(Parked_t) = \langle k,\, k-1 \rangle.$$

Third, if $1-p-q = -1$, then $p+q = 2$, $p = q = 1$, and we obtain $\mathbf{P}(Parked_t) = \langle k,\, 1-k \rangle$ for even values of t and $\mathbf{P}(Parked_t) = \langle 1-k,\, k \rangle$ for odd values of t. Thus $\mathbf{P}(Parked_t)$ converges only if $k = 1-k = \frac{1}{2}$ and $\lim_{t \to \infty} \mathbf{P}(Parked_t) = \langle \frac{1}{2},\, \frac{1}{2} \rangle$.