PROBABILISTIC REASONING OVER TIME

Outline

- Time and uncertainty
- Inference in temporal models
- Hidden Markov models
- Dynamic Bayesian networks

Based on the textbook by Stuart Russell & Peter Norvig: Artificial Intelligence, Modern Approach (2nd Edition)

Chapter 15; excluding Sections 15.4 and 15.6

TIME AND UNCERTAINTY

- We have previously developed our techniques for probabilistic reasoning in the context of static worlds.
- E.g. when repairing a car, it is assumed that whatever is broken remains broken during the process of diagnosis.
- However, in certain domains dynamic aspects become essential.

Example. A doctor is treating a diabetic patient,

- Recent insulin doses, food intake, blood sugar measurements, and other physical signs serve as pieces of evidence,
- The doctor decides about food intake and insulin dose.

States and Observations

- The process of change is viewed as a series of snapshots, each of which describes the state of the world at a particular time.
- Each time slice involves a set of random variables indexed by \( t \):
  1. the set of unobservable state variables \( X_t \)
  2. the set of observable evidence variables \( E_t \).

- The observation at time \( t \) is \( E_t = e_t \), for some set of values \( e_t \).

- The notation \( X_{a:b} \) denotes the set of variables from \( X_a \) to \( X_b \).

- The interval between time slices depends on the problem!

Stationary Processes and the Markov Assumption

- In a stationary process, the changes in the world state are governed by laws that do not themselves change over time.

- A first-order Markov process satisfies an equation

\[
P(X_t \mid X_{t-1}) = P(X_t \mid X_{t-1})
\]

where \( P(X_t \mid X_{t-1}) \) forms the transition model of the process.

- In addition, it is typical to assume a sensor model of the form

\[
P(E_t \mid X_{0:t}, E_{0:t-1}) = P(E_t \mid X_t)
\]

so that observations depend only on the current state.
**Example.** A security guard is working at some secret underground installation and would like to know whether it is raining today or not. The only access to the outside world occurs each morning when the director comes in, with or without, an umbrella.

- The set of state variables $X_t = \{\text{Rain}_t\}$ for $t = 0, 1, \ldots$.
- The set of evidence variables $E_t = \{\text{Umbrella}_t\}$ for $t = 1, 2, \ldots$.

<table>
<thead>
<tr>
<th>$R_{t-1}$</th>
<th>$P(R_t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
<td>0.7</td>
</tr>
<tr>
<td>$j$</td>
<td>0.3</td>
</tr>
</tbody>
</table>

\[ \text{Rain}_{t-1} \quad \text{Rain}_t \quad \text{Rain}_{t+1} \]
\[ \text{Umbrella}_{t-1} \quad \text{Umbrella}_t \quad \text{Umbrella}_{t+1} \]

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**Inference in Temporal Models**

Having set up the generic temporal model, we may formulate the basic inference tasks that are to be solved.

- **Filtering** or **monitoring**: the task is to compute the belief state, i.e., the posterior distribution $P(X_t \mid e_{1:t})$ over the current state.
- **Prediction**: the posterior distribution $P(X_{t+k} \mid e_{1:t})$ over the future state is of interest for some $k > 0$.
- **Smoothing** or **hindsight**: the aim is to compute $P(X_k \mid e_{1:t})$ where $0 \leq k < t$ for some past state.
- **Most likely explanation**: is a sequence of states $x_{1:t}$ that maximizes $P(x_{1:t} \mid e_{1:t})$ for the observations to date.

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**Resulting Joint Distribution**

- In addition to transition and sensor models, we need to specify a prior distribution $P(X_0)$ over the state at time 0.
- Combining this with the preceding transition and sensor models, which are independence assumptions, implies a distribution

\[ P(X_{0:t}, E_t) = P(X_0) \prod_{i=1}^t P(X_i \mid X_{i-1})P(E_i \mid X_i). \]

for any point of time $t$.

- If necessary, the Markov assumption can be recovered by introducing suitable state variables.

**Example.** When modeling a battery-powered robot wandering in the $xy$ plane, the battery level has to be taken into account.

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**Filtering and prediction**

- In **recursive estimation**, the idea is to compute $P(X_{t+1} \mid e_{1:t+1})$ as a function of $e_{t+1}$ and $P(X_t \mid e_{1:t})$.

- Using transition and sensor models we obtain by conditioning that

\[ P(X_{t+1} \mid e_{1:t+1}) = \alpha P(e_{t+1} \mid X_{t+1})P(X_{t+1} \mid e_{1:t}) \]

\[ = \alpha P(e_{t+1} \mid X_{t+1}) \sum_{x_{t+1}} P(X_{t+1} \mid x_t)P(x_t \mid e_{1:t}). \]

- This can be viewed viewed as the propagation of a message $f_{t, t+1} = P(X_t \mid e_{1:t})$ forward: $f_{t, t+1} = \alpha \text{FORWARD}(f_{t+1, t}, e_{t+1})$.

- **Prediction** is filtering without the addition of new evidence:

\[ P(X_{t+k} \mid e_{1:t}) = \sum_{x_{t+k}} P(X_{t+k} \mid x_{t+k})P(x_{t+k} \mid e_{1:t}). \]
Example. The security guard has a prior belief \( P(R_0) = (0.5, 0.5) \) about the state,

1. The prediction from \( t = 0 \) to \( t = 1 \) gives
   \[
   P(R_1) = \sum_0 P(R_1 | R_0)P(R_0)
   = (0.7, 0.3) \times 0.5 + (0.3, 0.7) \times 0.5 = (0.5, 0.5).
   \]
2. Updating this distribution with the evidence \( u_1 \) for \( t = 1 \) gives
   \[
   P(R_1 | u_1) = \alpha P(u_1 | R_1)P(R_1) = (0.9, 0.2)(0.5, 0.5)
   = \alpha(0.45, 0.1) \approx (0.818, 0.182).
   \]
3. Similarly, we obtain
   \[
   P(R_2 | u_1, u_2) = \alpha(0.565, 0.075) \approx (0.883, 0.117).
   \]
   The probability of rain increases due to repeated evidence.

Example. Let us demonstrate smoothing with the umbrella example:

1. \( P(R_1 | u_1, u_2) = \alpha f_{11} b_{22} = \alpha P(R_1 | u_1)P(u_2 | R_1) \) where we already know the distribution \( f_{11} = P(R_1 | u_1) = (0.818, 0.182). \)
2. The distribution \( b_{22} = P(u_2 | R_1) = \sum_2 P(u_2 | r_2)P(r_2 | R_1) = 0.9 \times (0.7, 0.3) + 0.2 \times (0.3, 0.7) = (0.69, 0.41). \)
3. By substituting these distributions and normalizing, we obtain
   \[
   P(R_1 | u_1, u_2) = \alpha(0.818, 0.182)(0.69, 0.41) \approx (0.883, 0.117).
   \]
   Thus the smoothed estimate is higher than the filtered estimate.

The additional piece of evidence \( u_2 \) increases the probability of rain on the first day, as the rain tends to persist.

### Smoothing

- The task is to compute \( P(X_k | e_{1:t}) \) for \( 0 \leq k < t \) referring to past,
- Using a backward message \( b_{k+1,t} = P(e_{k+1,t} | X_t) \), we obtain
  \[
  P(X_k | e_{1:t}) = \alpha f_{1k} b_{k+1,t}.
  \]
- The backward message \( b_{k+1,t} \) can be computed using
  \[
  b_{k+1,t} = \sum_{x_{k+1}} P(e_{k+1} | x_{k+1})P(e_{k+2:t} | x_{k+1})P(x_{k+1} | X_k).
  \]
- Whenever \( k + 1 = t \), the sequence \( e_{k+2:t} \) becomes empty and
  \[
  P(e_{k+2:t} | x_{k+1}) = P(\top | x_{k+1}) = 1 \text{ where } \top \text{ stands for truth},
  \]
- This leads to a recursive definition, or algorithm
  \[
  b_{k+1,t} = \alpha \text{BACKWARD}(b_{k+2:t}, e_{k+1:t}).
  \]

### Finding the Most Likely Sequence

Example. Suppose that the security guard makes the following observations during the first five days: \( u_1, u_2, \neg u_3, u_4, u_5 \).

What is the weather sequence most likely to explain this?

- For each pair of states \( x_{i+1} \) and \( x_i \), there is a recursive relationship between the most likely paths to \( x_{i+1} \) and \( x_i \).
- This is analogous to filtering, but the forward message \( f_{1:i} = P(X_i | e_{1:i}) \) must be replaced by
  \[
  m_{1:i} = \max_{X_{i-1}} P(x_1, \ldots, x_{i-1}, x_i | e_{1:i})
  \]
  and summation over \( x_i \) is replaced by maximization over \( x_i \).
- This gives the essential content of the Viterbi algorithm which has both linear time and space requirements.
**HIDDEN MARKOV MODELS**

- In a **hidden Markov Model** (HMM), the world is described by a single discrete random variable $X_i$ taking values $1, \ldots, S$ which correspond to the states of the world.
- The transition model $P(X_i \mid X_{i-1})$ becomes an $S \times S$ matrix $T$ such that $T_{ij} = P(X_i = j \mid X_{i-1} = i)$.
- Forward and backward reasoning are simplified as follows:
  $f_{i+1} = \alpha_{i+1} T f_i$
  $b_{i+1} = \alpha T b_{i+1} b_{i+2}$
  where $O_i$ is a diagonal matrix having $P(e_i \mid X_i = i)$ as the $i$th value.
- For HMMs, the time and space complexities of forward backward type reasoning are of orders $S^2 \times t$ and $S \times t$, respectively.

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**DYNAMIC BAYESIAN NETWORKS**

- A **dynamic Bayesian network** (DBN) represents how the state of the environment evolves over time.
- Each time slice of a DBN may have any number of state variables $X_i$ and evidence variables $E_i$.
- Every HMM can be transformed into a DBN and vice versa.
- By decomposing the state of a complex system into its constituent variables, the DBN is able to take advantage of the sparseness in the temporal probability model.

**Example.** The transition model of a DBN with 20 Boolean state variables, each of which has three parents in the preceding slide, has $20 \times 2^3 = 160$ probabilities while its HMM counterpart has $2^{40}$.

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**Constructing Dynamic Bayesian Networks**

- To construct a DBN, one must specify three distributions: $P(X_0)$, the transition model $P(X_{i+1} \mid X_i)$, and the sensor model $P(E_i \mid X_i)$.
- For each time step $t$, there is one node for each state variable $X_i$ and each evidence variable $E_i$ plus relevant links between nodes.

**Example.** For the security guard example, it is sufficient to specify

$$
\begin{array}{c}
\text{Rain}_0 \\
\text{Rain}_1 \\
\text{Umbrella}_1
\end{array}
$$

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**Example.** A robot is described with state variables $X_i = (X_i, Y_i)$ for position and $X_i = (X_i, Y_i)$ for velocity and $Battery_i$ for the actual battery charge level.

$$
\begin{array}{c}
\text{Battery}_0 \\
\text{Battery}_1
\end{array}
$$

Both position (evidence variables $Z_i$) and the battery charge level (evidence variable $BMeter_i$) are measured.
**Exact Inference in DBNs**

- The previous algorithms for inference in Bayesian networks can be applied to dynamic Bayesian networks.
- Given a sequence of observations, one can **unroll** a DBN until the network is large enough to accommodate the observations.
- Unrolling can also be done on a slice-by-slice basis.
- In the general case, the complexity of reasoning is exponential.

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**SUMMARY**

- A dynamic world can be handled using a set of random variables to represent the state of the world at each point in time.
- Representations can be designed to satisfy the **Markov property**, so that the future is independent of the past given the present.
- Combined with the **stationarity** assumption much simpler probabilistic models are obtained.
- A temporal probability model consists of a **transition model** and a **sensor model**.