# MAKING COMPLEX DECISIONS

**Outline**

- Sequential Decision Problems
- Value Iteration
- Policy Iteration
- Decision-Theoretic Agent Design
- Dynamic Belief/Decision Networks

Based on the textbook by S. Russell & P. Norvig:

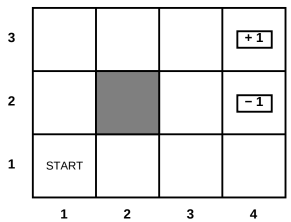
Artificial Intelligence, A Modern Approach, Chapter 17





# SEQUENTIAL DECISION PROBLEMS

**Example.** Consider an agent situated in the following environment:




- The agent may perform actions $North$, $South$, $East$, and $West$ in order to move between squares (or states) $(1,1)$, ..., $(4,3)$.
- Moving towards a wall results in no change in position.
- The operation of the agent stops and it receives a *reward/ punishment* if it reaches a square marked with $+1/-1$.





# Transition Model

- In a *deterministic setting* the outcomes of actions are known, and the agent may **plan** a sequence of actions which moves it to $(4,3)$.
- This becomes impossible if actions are *nondeterministic/unreliable*.
- A **transition model** assigns a probability $M^a_{ij}$ to the event that the agent reaches state $j$ it performs action $a$ in state $i$.

**Example.** (Continued) Each one of the four actions $North$, $South$, $East$, and $West$ moves the agent

1. to the intended direction $d$ with a probability of $0.8$, and
2. at right angles to the direction $d$ with probabilities $0.1$ and $0.1$.

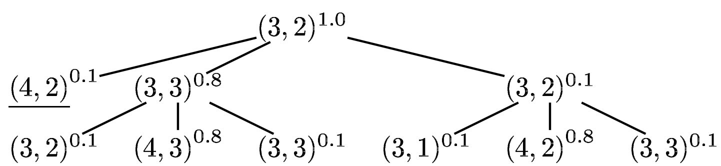




**Example.** If an action sequence $S = [North, East]$ is performed in state $(3,2)$ the agent reaches states with folowing probabilities:

$$\begin{aligned}
P_{(3,1)} &= 0.1 \times 0.1 = & 0.01\\
P_{(3,2)} &= 0.8 \times 0.1 = & 0.08\\
P_{(3,3)} &= 0.8 \times 0.1 + 0.1 \times 0.1 = & 0.09\\
P_{(4,2)} &= 0.1 + 0.1 \times 0.8 = & 0.18\\
P_{(4,3)} &= 0.8 \times 0.8 = & 0.64\\
\hline
& & 1.00
\end{aligned}$$

These are easily inspected from a (partial) *reachability graph*:



## Assigning Utilities to Plans?

- Utility function $U$ is based on a sequence of states (an **environment history**) rather than a single state.

**Example.** In our example, the utility is defined as the value of the terminal state ($+1$ or $-1$) minus $\frac{1}{25}$ of the length of the sequence.

- Considering sequences of actions as *long actions* implies committing to an entire sequence of actions before executing it.
- In practice, the agent should be able to choose a new action in each state *given any additional information provided by sensors*.
- In stochastic environments, plans have to be conditional and it may be impossible to set a limit for lengths of conditional plans.





## Policies

- We concentrate on **accessible** environments where the agent's percepts are always sufficient for determining the state it is in.
- A **policy** is a complete mapping from states to actions.
- Given a policy, it is possible to calculate the expected utility of the possible environment histories generated by that policy.
- It is non-trivial to compute an **optimal policy** that results in the highest expected utility (recall the MEU principle).
- If the agents knows an optimal policy, then it can choose an action in a deterministic fashion in every state.





**Example.** An optimal policy for the square world appears on the left.

| | | | | |
|---|---|---|---|---|
| 3 | → | → | → | +1 |
| 2 | ↑ | | ↑ | −1 |
| 1 | ↑ | ← | ← | ← |
| | 1 | 2 | 3 | 4 |

| | | | | |
|---|---|---|---|---|
| 3 | 0.812 | 0.868 | 0.918 | +1 |
| 2 | 0.762 | | 0.660 | −1 |
| 1 | 0.705 | 0.655 | 0.611 | 0.388 |
| | 1 | 2 | 3 | 4 |

The expected utilities for individual states are given on the right.

- The policy is very conservative (tries to avoid punishment).
- If the cost of moves is increased, then the optimal policy becomes different for the state $(3, 1)$: $West$ is replaced by $North$.
- If the cost of moves is decreased to $\frac{1}{100}$, then $West$ is chosen instead of $North$ in state $(3, 2)$.





## Markov Decision Problems

- The problem of calculating an optimal policy in an accessible, stochastic environment with a known transition model is called a **Markov decision problem** (MDP).
- It is said that the **Markov property** holds if the transition probabilities depend only on the state (not on previous history).
- In the sequel, we will study two basic techniques for solving MDPs, namely **value iteration** and **policy iteration**.
- In an inaccessible environment, the corresponding problem is called a **partially observable** MDP (or POMDP).
- Solving POMDPs is much more difficult than solving MDPs.

## VALUE ITERATION

- In **value iteration**, the idea is to compute the utility $U(s)$ for each state $s$ and to use these utilities for selecting optimal actions.
- It is difficult to determine $U(s)$ because of uncertain actions.
- Let $H(s,p)$ denote the history tree which results when – starting from a state $s$ – actions are taken according to a policy $p$.
- Given a transition model $M$, the expected utility of a state $s$ is

$$\begin{aligned} U(s) = & \ \mathrm{EU}(H(s, \mathit{policy}^*) \mid M) \\ = & \ \sum P(H(s, \mathit{policy}^*) \mid M) U_h(H(s, \mathit{policy}^*)) \end{aligned}$$

where $\mathit{policy}^*$ is an optimal policy defined by $M$ and the utility function $U_h$ on state histories.





## How to Derive an Optimal Policy?

- It is required that the utility function $U_h$ on histories is **separable**:

$$U_h([s_0, \ldots, s_n]) = f(s_0, U_h([s_1, \ldots, s_n])) \text{ for some } f.$$

- The simplest form of a separable utility function is **additive**:

$$U_h([s_0, s_1, \ldots, s_n]) = R(s_0) + U_h([s_1, \ldots, s_n])$$

where $R$ is a **reward function** on individual states $s$.

- Given an additive utility function $U_h$, an optimal policy $\mathit{policy}^*$ in state $i$ can be defined by the standard MEU principle:

$$\mathit{policy}^*(i) = \arg\ \max_a \sum_j M_{ij}^a U(j).$$

- Similarly, the utility of a state can be expressed as folows:

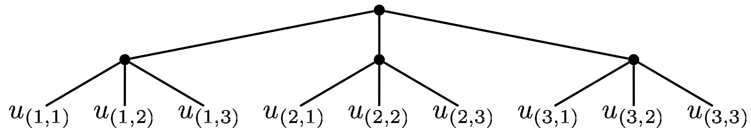
$$U(i) = R(i) + \max_a \sum_j M_{ij}^a U(j).$$





## Dynamic Programming

- Dynamic programming involves an $n$-step decision problem where the terminal states reached after $n$ steps have known utilities.




- The expected utilities of other states can be computed backwards (layer by layer): $n-1^{\text{th}}$ layer, $n-2^{\text{th}}$ layer, etc.
- In this fashion, the time complexity of computing utilities is of $O(n|A||S|)$ where $|S|$ is the number of reachable states.
- Unfortunately, the dynamic programming approach is no longer applicable if environment histories are of unbounded length.





## Value Iteration Algorithm

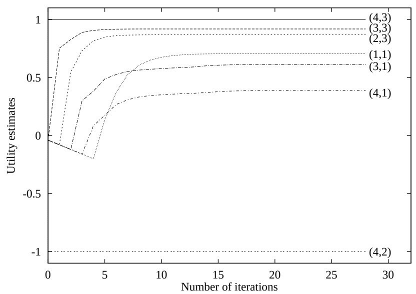
- There is an iterative procedure that approximates the utilities of states to any degree of accuracy.
- The next estimate $U_{t+1}(i)$ is based on the old utility estimates of the neighboring states: $U_{t+1}(i) = R(i) + \max_a \sum_j M_{ij}^a U_t(j)$.
- There is no bound on the length of actions sequences.

```
function VALUE-ITERATION(M, R) returns a utility function
  inputs: M, a transition model
          R, a reward function on states
  local variables: U, utility function, initially identical to R
                   U', utility function, initially identical to R

  repeat
     U ← U'
     for each state i do
        U'[i] ← R[i] + max_a Σ_j M^a_ij U[j]
     end
  until CLOSE-ENOUGH(U, U')
  return U
```

## Convergence

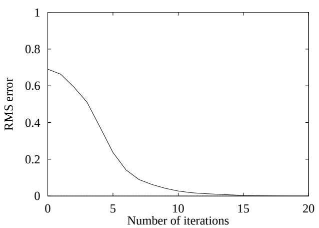
- As $t$ grows, the utility values will converge to stable values given certain conditions on the environment.
- Given a stabilized utility function, the corresponding *optimal policy* [shown by Bellman and Dreyfus, 1962] is easy to compute.

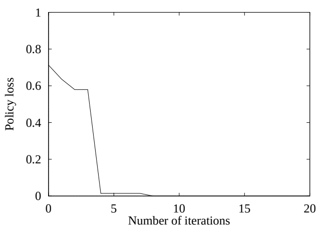







- Unfortunately, it is difficult to estimate how long the value iteration algorithm should be run to get an optimal policy.
- The progress of value iteration can be measured using **root mean square error** (RMS error) *if the correct values are known*.
- Alternatively, policies can be evaluated using **policy loss**, i.e., the difference of expected utility with respect to the optimal policy.




☞ An optimal policy is reached long before utilities converge.





## POLICY ITERATION

- The optimal policy is often not very sensitive to the utility values.
- The basic idea in **policy iteration** is to choose a policy $p$, calculate utilities using $p$ as policy, and update $p$ (repeatedly).
- The **value determination** (utilities) is simpler given a policy $p$:

$$U_{t+1}(i) = R(i) + \sum_j M_{ij}^{p(i)} U_t(j).$$

☞ A modified value iteration algorithm can be used.

- Unfortunately, value iteration may converge very slowly.
- Another approach is to solve utilities directly using equations

$$U(i) = R(i) + \sum_j M_{ij}^{p(i)} U(j)$$

that characterize stabilized utility values ($\forall i$: $U_{t+1}(i) = U_t(i)$).



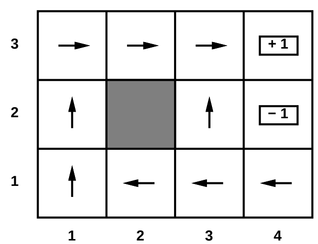


**Example.** The utilities of states $(3,2)$ and $(3,3)$ are solved as folows:

$$\begin{cases} u_{(3,2)} = -0.04 + 0.8u_{(3,3)} + 0.1u_{(3,2)} - 0.1 \\ u_{(3,3)} = -0.04 + 0.8 + 0.1u_{(3,3)} + 0.1u_{(3,2)} \end{cases}$$

$$\Longrightarrow \begin{cases} -0.8u_{(3,3)} = -0.9u_{(3,2)} - 0.14 \\ 8.1u_{(3,3)} = 0.9u_{(3,2)} + 6.84 \end{cases}$$

$$\Longrightarrow u_{(3,3)} = \frac{6.7}{7.3} \approx 0.918 \text{ and } u_{(3,2)} = \frac{0.8u_{(3,3)} - 0.14}{0.9} \approx 0.660.$$




| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 3 | 0.812 | 0.868 | 0.918 | +1 |
| 2 | 0.762 | | 0.660 | −1 |
| 1 | 0.705 | 0.655 | 0.611 | 0.388 |

- Once the utilities of all states are known, it is straightforward to update the current policy using the MEU principle.

**function** POLICY-ITERATION($M, R$) **returns** a policy
  **inputs**: $M$, a transition model
    $R$, a reward function on states
  **local variables**: $U$, a utility function, initially identical to $R$
    $P$, a policy, initially optimal with respect to $U$

  **repeat**
    $U \leftarrow$ VALUE-DETERMINATION($P, U, M, R$)
    *unchanged?* ← true
    **for each** state $i$ **do**
      **if** $\max_a \sum_j M^a_{ij} U[j] > \sum_j M^{P[i]}_{ij} U[j]$ **then**
        $P[i] \leftarrow \arg\max_a \sum_j M^a_{ij} U[j]$
        *unchanged?* ← false
    **end**
  **until** *unchanged?*
  **return** $P$





## How Immortal Agents Decide What to Do?

- The total reward obtained by a policy can easily be unbounded if the lifetime of an agent is not limited.
- In **discounting**, rewards received in the future are considered less valuable than rewards received in the current time step.
- Given a **discount factor** $0 \leq \gamma < 1$, the sum $U(H) = \sum_{i=1}^{\infty} \gamma^i R_i$ of rewards $R_1, R_2, \ldots$ (bounded by $R$) in a history $H$ converges.
- Discounting conforms to a preference-independence assumption called **stationarity**: if $R_1 = S_1$ holds for two reward sequences $R_1, R_2, \ldots$ and $S_1, S_2, \ldots$, then these sequences should be preference ordered in the same way as $R_2, R_3, \ldots$ and $S_2, S_3, \ldots$.
- An optimal policy yields a constant **system gain** in the long run.





## DECISION-THEORETIC AGENT DESIGN

- Recall the schematic design of decision theoretic agents performing **decision cycles** repeatedly:

**function** DECISION-THEORETIC-AGENT( *percept*) **returns** *action*

  calculate updated probabilities for current state based on
    available evidence including current percept and previous action
  calculate outcome probabilities for actions
    given action descriptions and probabilities of current states
  select *action* with highest expected utility
    given probabilities of outcomes and utility information
  **return** *action*

- The components of the cycle are refined gradually in the sequel.
- We begin with the problem of determining the current state.





## Determining the Current State of the World

- In general, it is assumed that a set of state variables $\mathbf{X}_t$ (indexed by time $t$) refers to the current state of the world.
- Given the percept history $\mathbf{E}_1, \ldots, \mathbf{E}_t$ and the previous actions $A_1, \ldots, A_{t-1}$, we are interested in the probability distribution

$$Bel(\mathbf{X}_t) = \mathbf{P}(\mathbf{X}_t | \mathbf{E}_1, \ldots, \mathbf{E}_t, A_1, \ldots, A_{t-1}).$$

- The direct evaluation of $Bel(\mathbf{X}_t)$ is out of the question, as it requires conditioning on many variables.
- Conditional independence statements can be introduced in order to simplify the expression for $Bel(\mathbf{X}_t)$.

## Simplifying Assumptions

- Assuming the Markov property, we obtain

$$\mathbf{P}(\mathbf{X}_t \mid \mathbf{X}_1, \ldots, \mathbf{X}_{t-1}, A_1, \ldots, A_{t-1}) = \mathbf{P}(\mathbf{X}_t \mid \mathbf{X}_{t-1}, A_{t-1}).$$

- The Markov property can be established by introducing state variables that record relevant information from percepts.

  **Example.** If the robot is battery-powered, then the state variable $BatteryLevel_t$ is needed to restore the Markov property.

- Percepts are causally determined by the state of the world:

$$\mathbf{P}(\mathbf{E}_t|\mathbf{X}_1, \ldots, \mathbf{X}_t, A_1, \ldots, A_{t-1}, \mathbf{E}_1, \ldots, \mathbf{E}_{t-1}) = \mathbf{P}(\mathbf{E}_t|\mathbf{X}_t).$$

- The action taken depends only on the percepts received to date:

$$\mathbf{P}(A_{t-1}|A_1, \ldots, A_{t-2}, \mathbf{E}_1, \ldots, \mathbf{E}_{t-1}) = \mathbf{P}(A_{t-1}|\mathbf{E}_1, \ldots, \mathbf{E}_{t-1}).$$





## Calculating the State Estimate $Bel(\mathbf{X}_t)$

- The calculation takes place in two phases:
  1. **Prediction phase:** the prior probability distribution $\widehat{Bel}(\mathbf{X}_t)$ based on the previous state and how actions affect states:

$$\sum_{\mathbf{x}_{t-1}} \mathbf{P}(\mathbf{X}_t \mid \mathbf{X}_{t-1} = \mathbf{x}_{t-1}, A_{t-1}) Bel(\mathbf{X}_{t-1} = \mathbf{x}_{t-1}).$$

  2. **Estimation phase:** the effect of the most recent percept $\mathbf{E}_t$ is incorporated to the distribution $\widehat{Bel}(\mathbf{X}_t)$ by Bayesian updating:

$$Bel(\mathbf{X}_t) = \alpha \mathbf{P}(\mathbf{E}_t \mid \mathbf{X}_t) \widehat{Bel}(\mathbf{X}_t)$$

  where $\alpha$ is a normalization constant.

- The equations for $Bel$ and $\widehat{Bel}$ form a generalization of **Kalman filtering** – a technique of classical control theory.





## The Complete Decision-Theoretic Design

- The remaining steps of the decision cycle are straightforward.

**function** DECISION-THEORETIC-AGENT($\boldsymbol{E}_t$) **returns** an action
  **inputs**: $\boldsymbol{E}_t$, the percept at time $t$
  **static**: $BN$, a belief network with nodes $\mathbf{X}$
        $Bel(\boldsymbol{X})$, a vector of probabilities, updated over time

  $\widehat{Bel}(\mathbf{X}_t) \leftarrow \sum_{\mathbf{X}_{t-1}} \mathbf{P}(\mathbf{X}_t \mid \mathbf{X}_{t-1}{=}\mathbf{x}_{t-1}, A_{t-1})\ Bel(\mathbf{X}_{t-1}{=}\mathbf{x}_{t-1})$
  $Bel(\mathbf{X}_t) \leftarrow \alpha\ \mathbf{P}(\mathbf{E}_t \mid \mathbf{X}_t)\ \ \widehat{Bel}(\mathbf{X}_t)$
  $action \leftarrow \arg\max_{A_t} \sum_{\mathbf{X}_t} \left[ Bel(\mathbf{X}_t{=}\mathbf{x}_t) \sum_{\mathbf{X}_{t+1}} P(\mathbf{X}_{t+1} = \mathbf{x}_{t+1} \mid \mathbf{X}_t{=}\mathbf{x}_t, A_t)\ U(\mathbf{x}_{t+1}) \right]$
  **return** *action*

- The **sensor model** $\mathbf{P}(\mathbf{E}_t \mid \mathbf{X}_t)$ describes how the environment generates the sensor data.
- The **action model** $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1}, A_{t-1})$ gives the effects of actions.





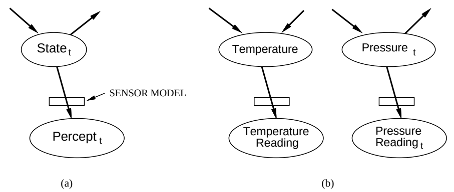
## Sensing in Uncertain Wods

- A sensor model is **stationary** if it holds for all $t$ that

$$\mathbf{P}(\mathbf{E}_t \mid \mathbf{X}_t) = \mathbf{P}(\mathbf{E} \mid \mathbf{X}).$$

  ☞ A fixed model $\mathbf{P}(\mathbf{E} \mid \mathbf{X})$ can be used at each time step.

- All variables affecting the sensors have to be included in $\mathbf{X}$.
- A sensor model is easily implemented as a conditional probability table in a belief network (the figure on the left hand side):



- The values of sensors are *causally* related to the state of the world.
- A perfect sensor corresponds to a purely deterministic CPT.
- Possible *noise* and *errors* in the sensor are taken into account in the probabilities of incorrect readings.

**Example.** In the burglar-alarm example, both $JohnCalls$ and $MaryCalls$ can be viewed as sensors for the $Alarm$ state variable.

- Typically, each sensor only measures some small aspects of the total state (as illustrated in the figure on the right hand side).
- Decomposing the overall sensor model into several components may reduce the size of the CPTs required.

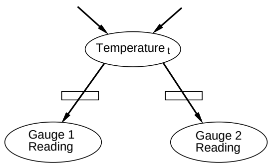
**Example.** Measuring $Pressure$ and $Temperature$ with sensors that measure $Pressure/Temperature$ and $Pressure \times Temperature$ leads to complicated sensor models that depend on both state variables.





## Sensor Fusion

- There are often several sensors measuring the same state variable.




- The sensor values are conditionally independent of each other, given the actual value of the state variable.
- **Sensor fusion** or **data fusion** is about interpreting and putting together perceptual information from multiple sensors.

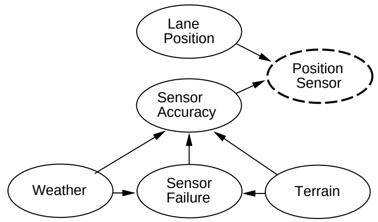
**Example.** If the readings from gauges are $13.6°K(\pm 0.5°K)$ and $14.4°K(\pm 0.5°K)$, the temperature is between $13.9°K$ and $14.1°K$.





## Sensor Failures

- It may be difficult to detect a sensor failure.
- To handle sensor failures in the first place the possibility of failure has to be taken into account in the sensor model.
- Sensor fusion may discount the readings of a failed sensor.
- One possibility is to add a detailed sensor failure model:





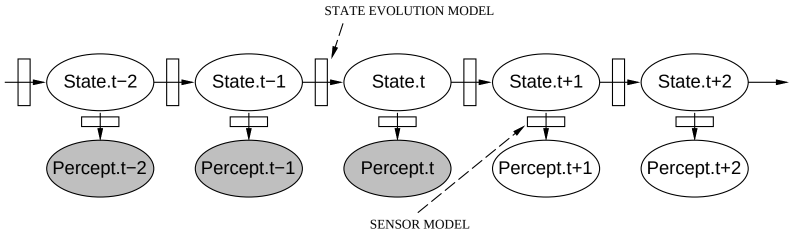



## DYNAMIC BELIEF NETWORKS

- A **dynamic belief network** (DBN) represents how the state of the environment evolves over time.
- In analogy to sensor models, a **stationarity** assumption is made: the distribution $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1}, A_{t-1})$ is the same for all $t$.
- Moreover, the agent is assumed to be passively monitoring and predicting a changing environment (i.e., it performs no actions).
- A **state evolution model** where a sequence of $\mathbf{X}_t$ values is based on a fixed distribution $\mathbf{P}(\mathbf{X}_t \mid \mathbf{X}_{t-1})$ is called a **Markov chain**.
- DBNs will be generalized for the decision of actions later on.

## Structure of Dynamic Belief Networks

- For each time step $t$, there is one node for each state variable $X_t$ and sensor variable $Y_t$ – including appropriate interconnections.




- The task is to calculate the probability distribution for $State_t$, given the evidence for $\ldots, Percept_{t-1}, Percept_t$.
- **Probabilistic projection** means estimating how the state of the environment (i.e., $State_{t+n}$ with $n > 0$) evolves in the future.





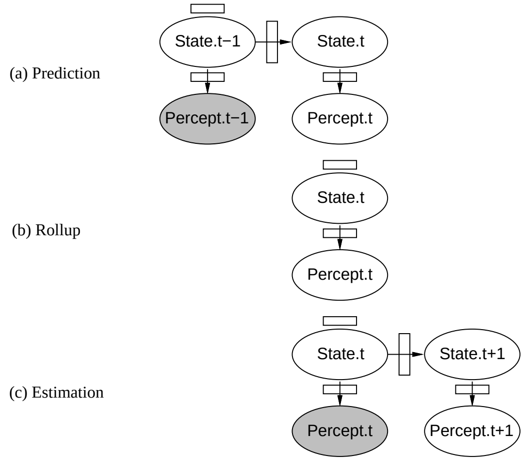
## Prediction and Estimation with Belief Networks

- The prediction and estimation phases of the refined decision cycle can be implemented as operations on belief networks.
- It is sufficient to consider two time steps $t$ and $t-1$ (also called the **slices** of the network):
  (a) **Prediction:** using $Bel(\mathbf{X}_{t-1})$ calculate the prior distribution $\widehat{Bel}(\mathbf{X}_t) = \sum_{\mathbf{X}_{t-1}} \mathbf{P}(\mathbf{X}_t \mid \mathbf{X}_{t-1} = \mathbf{x}_{t-1}) Bel(\mathbf{X}_{t-1} = \mathbf{x}_{t-1})$.
  (b) **Rollup:** remove the slice for $t-1$ from the network and add the prior probability tables (based on $\widehat{Bel}(\mathbf{X}_t)$) for $\mathbf{X}_t$.
  (c) **Estimation:** add the new percept $\mathbf{E}_t$, calculate $Bel(\mathbf{X}_t)$ by updating the network, and add the slice for $t+1$.
- After these three steps, the network is ready for the next cycle.





- Let us illustrate the three steps for prediction and estimation:

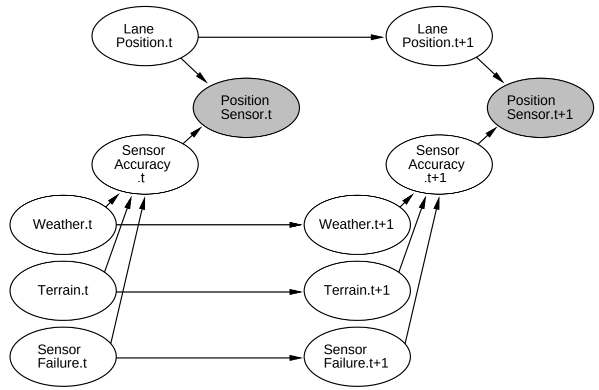



- Moreover, probabilistic projection is possible by adding the respective slices (*but without percept nodes*) for future time steps.





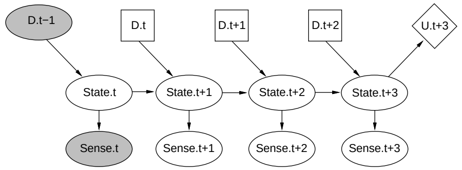
**Example.** Let us extend the sensor failure model presented earlier by adding state evolution models for the state variables $Weather$, $Terrain$, $SensorFailure$ and $LanePosition$:




- The model for the variable $SensorFailure$ determines that the sensor usually stays broken once it gets broken.

## DYNAMIC DECISION NETWORKS

- Any dynamic belief network can be converted into a **dynamic decision network** by adding utility nodes and decision nodes:




- The goal is to calculate the value of $D_t$ by the MEU principle.
- The utility of a decision sequence $\vec{d}$ is a weighted sum of utilities associated with each possible percept sequence given $\vec{d}$.
  The probabilities of percept sequences given $\vec{d}$ are used as weights.
- Dynamic decision networks solve POMDPs only approximately.





## Discussion

DDNs provide solutions to many problems arising in AI systems:

- Uncertainty is handled correctly, and sometimes efficiently.
- Continuous streams of sensor input can be dealt with.
- Unexpected events are supported, since fixed plans are not used.
- Noisy and failing sensors can be modeled.
- The relevance of information can be estimated before acquisition.
- Relatively large state spaces can be handled if states can be represented by state variables with sparse connections.
- There are techniques for approximative reasoning.





## SUMMARY

- A **optimal policy** associates an optimal decision with every state that the agent might reach.
- **Value iteration** and **policy iteration** are two methods for calculating optimal policies.
- Unbounded action sequences can be dealt with **discounting**.
- **Dynamic belief networks** can handle sensing and updating over time, and provide a direct implementation of the update cycle.
- **Dynamic decision networks** can solve sequential decision problems arising for agents in complex, uncertain domains.





## QUESTIONS

1. Recall the belief network that you designed for representing the ball tracking mechanism of a soccer playing agent.
   - Is it possible to identify a state evolution model and a sensor model from your network?
   - If not, reconstruct the network by keeping these in mind.
2. Continue the analysis of soccer playing agents.
   - Can you identify other problems in this domain that can be considered as real sequential decision problems?
   - Try to formalize such a problem as a dynamic decision network.