T-79.230 **Kevät 2002**
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Laskuharjoitus 1
Solutions

1. Let the set of propositions be:

$$\begin{aligned} A = &\quad \text{“A person is ill”} \\ B = &\quad \text{“The test is positive”} \end{aligned}$$

Since only one person in 10000 has the illness, the prior probability $P(A) = 1/10000$.

The test accuracy is 99% so the following conditional probabilities hold:

$$\begin{aligned} P(B \mid A) &= 0.99 \\ P(\neg B \mid \neg A) &= 0.99 \\ P(B \mid \neg A) &= 0.01 \\ P(\neg B \mid A) &= 0.01 \end{aligned}$$

There are two ways how the test may be positive: either the person is ill and the test gives the correct result, or the person is not ill but the test gives the wrong result. Since the two cases are mutually exclusive:

$$P(B) = P(B \mid A)P(A) + P(B \mid \neg A)P(\neg A)$$

We then use Bayes theorem to compute the probability that a person is ill given that the test result is positive:

$$\begin{aligned} P(A \mid B) &= \frac{P(B \mid A)P(A)}{P(B)} \\ &= \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid \neg A)P(\neg A)} \\ &= \frac{0.99 \cdot \frac{1}{10000}}{0.99 \cdot \frac{1}{10000} + 0.01 \cdot (1 - \frac{1}{10000})} \\ &= 0.0098 \end{aligned}$$

So, the patient does not have to worry, yet, since the probability of an incorrect diagnosis is much higher than of the illness.

2. We start with the case there the only evidence for A is B:

$$P(A \mid B) = \frac{P(B \mid A)}{P(B)} P(A)$$

We add here the evidence C using Bayesian updates (textbook page 429), and simplify the equation:

$$\begin{aligned}
P(A \mid B, C) &= P(A \mid B)\frac{P(C \mid A \wedge B)}{P(C \mid B)} \\
&= P(A)\frac{P(B \mid A)}{P(B)}\frac{P(C \mid A \wedge B)}{P(C \mid B)} \\
&= \frac{P(A)P(B \mid A)P(C \mid A \wedge B)}{P(C \wedge B)} \\
&= \frac{P(A \wedge B)P(C \mid A \wedge B)}{P(C \wedge B)} \\
&= \frac{P(A \wedge B \wedge C)}{P(C \wedge B)}
\end{aligned}$$

Similarily, if we start with evidence C and add B to it, we get:

$$P(A \mid C) = \frac{P(C \mid A)}{P(C)}P(A)$$

and

$$\begin{aligned}
P(A \mid C, B) &= P(A \mid C)\frac{P(B \mid A \wedge C)}{P(B \mid C)} \\
&= P(A)\frac{P(C \mid A)}{P(C)}\frac{P(B \mid A \wedge C)}{P(B \mid C)} \\
&= \frac{P(A)P(C \mid A)P(B \mid A \wedge C)}{P(B \wedge C)} \\
&= \frac{P(A \wedge C)P(B \mid A \wedge C)}{P(B \wedge C)} \\
&= \frac{P(A \wedge C \wedge B)}{P(B \wedge C)}
\end{aligned}$$

Thus, the end result is the same no matter what evidence is applied first.

3. We use propositions:

$$\begin{aligned}
A = &\text{ "Prisoner } A \text{ is executed"} \\
B = &\text{ "Prisoner } B \text{ is executed"} \\
C = &\text{ "Prisoner } C \text{ is executed"}
\end{aligned}$$

The propositions are mutually exclusive and each has a prior probability of 1/3. We add the proposition:

$$D = \text{ "The guard tells } B \text{ that he survives"}$$

After this the first proposition splits into two different cases: one where the guard tells B and one where he tells C. Since the guard does not lie, the other two propositions are not modified. Now:

$$
\begin{aligned}
P(\{A, D, \neg B, \neg C\}) &= \frac{1}{6} \\
P(\{A, \neg D, \neg B, \neg C\}) &= \frac{1}{6} \\
P(\{\neg A, \neg D, B, \neg C\}) &= \frac{1}{3} \\
P(\{\neg A, D, \neg B, C\}) &= \frac{1}{3}
\end{aligned}
$$

The guard tells A that B survives, so he now can compute his changes using the equation:

$$
\begin{aligned}
P(A \mid D) &= \frac{P(D \mid A)P(A)}{P(D)} \\
&= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{3} + \frac{1}{6}} = \frac{1}{3}
\end{aligned}
$$

4.

$$
\begin{aligned}
P(B \rightarrow A) &= P(\neg B \vee A) \\
&= P(A) + P(\neg B) - P(A \wedge \neg B) \\
&= P(A \wedge B) + P(A \wedge \neg B) - P(A \wedge B \wedge A \wedge \neg B) + P(\neg B) - \\
&\quad\ P(A \wedge \neg B) \\
&= P(A \wedge B) + P(\neg B) \\
&= P(A \mid B)P(B) + P(\neg B)
\end{aligned}
$$

In the third line of the derivation we use the identity:

$$
A \equiv (A \wedge B) \vee (A \wedge \neg B).
$$

Additionally, $P(A \wedge B \wedge A \wedge \neg B) = 0$, because $P(B \wedge \neg B) = 0$.