Uncertain Knowledge and Reasoning

Outline

- Acting under uncertainty
- Decision-theoretic agent
- Probability
- Bayes' rule

Acting under Uncertainty

- Agents almost never have access to the whole truth about their environment
- Qualification problem:

  how to define the circumstances under which a given action is *guaranteed* to work.
- Rational decision

  depends on the relative importance of the various goals and the likelihood that, and degree to which, they will be achieved.

Handling uncertain knowledge

- Example: diagnostic knowledge
  1. $\forall p(Symptom(p, Toothache) \rightarrow Disease(p, Cavity))$
  2. $\forall p(Symptom(p, Toothache) \rightarrow$ $Disease(p, Cavity) \vee Disease(p, GumDisease) \vee$ $Disease(p, ImpactedWisdom) \vee \cdots)$
  3. $\forall p(Disease(p, Cavity) \rightarrow Symptom(p, Toothache))$
- Difficulties

  —Laziness

  —Theoretical ignorance

  —Practical ignorance

Probability theory

- Agent's knowledge: degree of belief
- Probability: assigns a degree of belief between 0 and 1 to sentences. **A way of summarizing the uncertainty.**
- Sentences either true or false.

  — A probability of 0.8 that cavity if toothache.

  — A probability of 0 $\mapsto$ false

  — A probability of 1 $\mapsto$ true
- Degree of belief vs. degree of truth

  Probability theory vs. fuzzy logic.

- The probability that an agent assigns to a proposition depends of the percepts (evidence)
- Analogous to logical consequence
- Prior/unconditional probability
- Posterior/conditional probability

Decision-theoretic agent

- Utility theory (preferences)
- Decision theory = probability theory + utility theory
  Maximum expected utility: An agent is rational iff it chooses the action that yields the highest expected utility, averaged over all the possible outcomes of the action.

```
function DT-AGENT(percept) returns an action
   static: a set probabilistic beliefs about the state of the world

   calculate updated probabilities for current state based on
        available evidence including current percept and previous action
   calculate outcome probabilities for actions,
        given action descriptions and probabilities of current states
   select action with highest expected utility
        given probabilities of outcomes and utility information
   return action
```


Probability

- Prior/unconditional probability: $P(Cavity) = 0.1$
- Random variables:
  $P(Weather = Sunny) = 0.7$ $\quad P(Cavity = true) = 0.1$
  $P(Weather = Rain) = 0.2$ $\quad P(Cavity = false) = 0.9$
  $P(Weather = Cloudy) = 0.08$
  $P(Weather = Snow) = 0.02$
- Probability distribution:
  $\mathbf{P}(Weather) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$
  $\mathbf{P}(Weather, Cavity)$: 4×2 table

Conditional/posterior probability

$P(Cavity|Toothache) = 0.8$

$$P(A|B) = \frac{P(A \wedge B)}{P(B)}$$

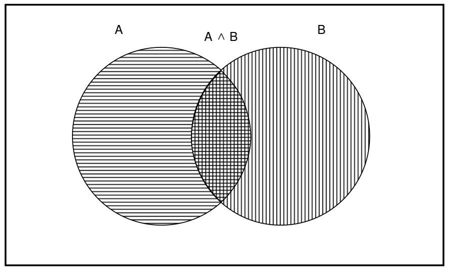
whenever $P(B) > 0$

- Product rule: $P(A \wedge B) = P(A|B)P(B)$
  $P(A \wedge B) = P(B|A)P(A)$
- $\mathbf{P}(X \wedge Y) = \mathbf{P}(X|Y)\mathbf{P}(Y)$
  tulkitaan yksittäisten arvojen suhteen:
  $P(X = x_1 \wedge Y = y_2) = P(X = x_1|Y = y_2)P(Y = y_2)$

Axioms of probability

1. $0 \leq P(A) \leq 1$ for all sentences A
2. $P(A) = 0/1$ if A is unsatisfiable/valid
3. $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$

True



- $P(A \wedge B) = P(A) + P(B) - P(A \vee B)$ (by 3)
- $P(\neg A) = 1 - P(A)$

$$\begin{aligned} P(A \vee \neg A) &= P(A) + P(\neg A) - P(A \wedge \neg A) \text{ (by 3)} \\ 1 &= P(A) + P(\neg A) - 0 \text{ (by 2)} \\ P(\neg A) &= 1 - P(A) \end{aligned}$$

Note that $A \vee \neg A$ is valid and $A \wedge \neg A$ is unsatisfiable.

Why the axioms of probability are reasonable

(Bruno de Finetti, 1931):
If Agent 1 expresses a set of degrees of belief that violate the axioms of probability theory then there is a betting strategy for Agent 2 that guarantees that Agent 1 will lose money.

Example.

Agent 1		Agent 2		Outcome for Agent 1			
Proposition	Belief	Bet	Stakes	$A \wedge B$	$A \wedge \neg B$	$\neg A \wedge B$	$\neg A \wedge \neg B$
A	0.4	A	4 to 6	-6	-6	4	4
B	0.3	B	3 to 7	-7	3	-7	3
$A \vee B$	0.8	$\neg(A \vee B)$	2 to 8	2	2	2	-8
				-11	-1	-1	-1

The joint probability distribution

- Atomic event: assignment of particular values to all the variables.
- The joint probability distribution $\mathbf{P}(X_1, \ldots, X_n)$: assigns probabilities to all possible atomic events.

	$Toothache$	$\neg Toothache$
$Cavity$	0.04	0.06
$\neg Cavity$	0.01	0.89

- Entries in the table sum to 1
- Can be used to compute any probabilistic statement

  $P(Cavity) = 0.04 + 0.06 = 0.10$

  $P(Cavity \lor Toothache) = 0.04 + 0.06 + 0.01 = 0.11$

$$\begin{aligned} P(Cavity|Toothache) &= \frac{P(Cavity \land Toothache)}{P(Toothache)} \\ &= \frac{0.04}{0.04 + 0.01} = 0.80 \end{aligned}$$

- For n Boolean variables, 2^n entries.

Bayes' Rule

- Bayes' rule is obtained from the product rule

  $(P(A \land B) = P(A|B)P(B),\ P(A \land B) = P(B|A)P(A))$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

- For multi-valued variables: $\mathbf{P}(Y|X) = \dfrac{\mathbf{P}(X|Y)\mathbf{P}(Y)}{\mathbf{P}(X)}$
- For background evidence E:

$$\mathbf{P}(Y|X,E) = \frac{\mathbf{P}(X|Y,E)\mathbf{P}(Y|E)}{\mathbf{P}(X|E)}$$

Applying Bayes's rule

$$\begin{aligned} P(S|M) &= 0.5 \\ P(M) &= 1/50000 \\ P(S) &= 1/20 \\ P(M|S) &= \frac{P(S|M)P(M)}{P(S)} \\ &= \frac{0.5 \times 1/50000}{1/20} = 0.0002 \end{aligned}$$

Diagnostic knowledge is often more tenuous than causal knowledge.

Normalization

How to avoid direct assessment of the prior probabilities of the "symptoms"?

$$\begin{aligned} P(M|S) &= \frac{P(S|M)P(M)}{P(S)} \\ P(\neg M|S) &= \frac{P(S|\neg M)P(\neg M)}{P(S)} \\ 1 &= P(M|S) + P(\neg M|S) \\ P(S) &= P(S|M)P(M) + P(S|\neg M)P(\neg M) \\ P(M|S) &= \frac{P(S|M)P(M)}{P(S|M)P(M) + P(S|\neg M)P(\neg M)} \end{aligned}$$

Combining evidence

$P(Cavity|Toothache) = 0.8$
$P(Cavity|Catch) = 0.95$

$$P(Cavity|Catch \wedge Toothache) = \frac{P(Catch \wedge Toothache|Cavity)P(Cavity)}{P(Catch \wedge Toothache)}$$

Many numbers have to be specified!
☞ Bayesian updating

$$P(Cavity|Toothache) = P(Cavity)\frac{P(Toothache|Cavity)}{P(Toothache)}$$

$P(Cavity|Toothache \wedge Catch) =$

$$P(Cavity|Toothache)\frac{P(Catch|Toothache \wedge Cavity)}{P(Catch|Toothache)} =$$

$$P(Cavity)\frac{P(Toothache|Cavity)}{P(Toothache)}\frac{P(Catch|Toothache \wedge Cavity)}{P(Catch|Toothache)}$$

Many numbers still! ☞
Conditional independence of $Tootache$ and $Catch$ given $Cavity$:
$P(Catch|Toothache \wedge Cavity) = P(Catch|Cavity)$
$P(Toothache|Catch \wedge Cavity) = P(Toothache|Cavity)$

- $P(Cavity|Toothache \wedge Catch) =$

  $$P(Cavity)\frac{P(Toothache|Cavity)}{P(Toothache)}\frac{P(Catch|Cavity)}{P(Catch|Toothache)}$$

- Provided we assess $P(Toothache|\neg Cavity)$ and $P(Catch|\neg Cavity)$

  $$P(Toothache)P(Catch|Toothache)$$

  can be eliminated by normalization.
- Conditional independence information crucial.

Where do the probabilities come from?

- Frequentist
- Objectivist
- Subjectivist
- Reference class problem