Some Incomplete Constraint Solvers

Pages: 178 – 196

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Objectives

- Introduce specialized techniques for some specific domains and constraints for which no efficient solving methods are known to exist
- Incomplete constraint solvers
  - Equality and disequality constraints
  - Boolean constraints
  - Linear constraints on integer intervals
    - Domain reduction rules for inequality constraints

Incomplete constraint solvers

- Equivalence preserving reduction rules
- If the constraints are defined using a specified language, the general rules of Chapter 5 cannot be used
- → the general framework is a customized to a specific language in which the constraints are defined and to specific domains that are used

Lemma 6.1 (Hyper-arc Consistency)

- Consider a hyper-arc consistent CSP *P*. Then *P* is closed under the application of every domain reduction rule which is
  - equivalence preserving, and
  - has only one constraint in its premise
- If a constraint solver imposes hyper-arc consistency, then it achieves the optimal domain reduction

Equality and Disequality Constraints

- $=, \neq$
- Arbitrary domains

Equality and Disequality Rules (1/2)

EQUALITY 1

$$\frac{\langle x = x \; ; \; x \in D\rangle}{\langle \; ; \; x \in D\rangle}$$

EQUALITY 2

$$\frac{\langle x = y \; ; \; x \in D_x, y \in D_y\rangle}{\langle x = y \; ; \; x \in D_x \cap D_y, y \in D_x \cap D_y\rangle}$$

Equality and Disequality Rules (2/2)

DISEQUALITY 1

$$\frac{\langle x \neq x \; ; \; x \in D\rangle}{\langle \; ; \; x \in \emptyset\rangle}$$

DISEQUALITY 2

$$\frac{\langle x \neq y \; ; \; x \in D_x, y \in D_y\rangle}{\langle \; ; \; x \in D_x, y \in D_y\rangle}$$

where $D_x \cap D_y = \emptyset$,

DISEQUALITY 3

$$\frac{\langle x \neq y \; ; \; x \in D, y = a\rangle}{\langle \; ; \; x \in D - \{a\}, y = a\rangle}$$

where $a \in D$

DISEQUALITY 4

$$\frac{\langle x \neq y \; ; \; x = a, y \in D\rangle}{\langle \; ; \; x = a, y \in D - \{a\}\rangle}$$

where $a \in D$

Proof System EQU

- **Theorem 6.2 (EQU)** A CSP with only equality and disequality constraints is hyper-arc consistent iff it is closed under the applications of the proof rules of the proof system EQU
- A constraint solver determined by the EQU rules is incomplete
- However are very simple operations on the variable domain and thus easy to implement

Boolean Constraints (1/2)

- **Boolean variables:** range over {0, 1}
- **Boolean domain expression:** $x \in D$ where $D \subseteq \{0,1\}$
- **Boolean expression:** built out of Boolean variables using the connectives: $\neg, \wedge, \vee$
- **Boolean constraint:** $s = t$, where s and t are Boolean expressions
- **Boolean CSP:** a CSP with *Boolean domain expressions* and *Boolean constraints*
- The predefined variables x_T and x_F represent the Boolean Constants ***true*** and ***false***

Boolean Constraints (2/2)

- **Simple** Boolean constraints:
  - **Equlity constraint:** $x = y$
  - **NOT constraint:** $\neg x = y$
  - **AND constraint:** $x \wedge y = z$
  - **OR constraint:** $x \vee y = z$
- **Compound** Boolean constraints can be reduced to simple Boolean constraints

Transformation Rules (1/2)

$$\frac{\langle \neg s = t; D \rangle}{\langle \neg x = t, s = x; D, x \in \{0,1\} \rangle}$$

where s is not a variable and x is a new variable

$$\frac{\langle \neg s = t; D \rangle}{\langle \neg s = y, t = y; D, y \in \{0,1\} \rangle}$$

where t is not a variable and y is a new variable

$$\frac{\langle s \quad op \quad t = u; D \rangle}{\langle s \quad op \quad t = z, u = z; D, x \in \{0,1\} \rangle}$$

where u is not a variable or a variable identical to s or t, x is a new variable and op is $\wedge$ or $\vee$

Transformation Rules (2/2)

$$\frac{\langle s \quad op \quad t = u; D \rangle}{\langle x \quad op \quad t = u, s = x; D, x \in \{0,1\} \rangle}$$

where s is not a variable or a variable identical to t or u, x is a new variable and op is $\wedge$ or $\vee$

$$\frac{\langle s \quad op \quad t = u; D \rangle}{\langle s \quad op \quad y = u, t = y; D, y \in \{0,1\} \rangle}$$

where t is not a variable or a variable identical to s or u, y is a new variable and op is $\wedge$ or $\vee$

Domain Reduction Rules

- For simple Boolean constraints

$$\frac{\langle x \wedge y = z; x = 1, y \in D_y, z \in D_z \rangle}{\langle y = z; x = 1, y \in D_y, z \in D_z \rangle}$$

Written as

$$x \wedge y = z, x = 1 \rightarrow y = z$$

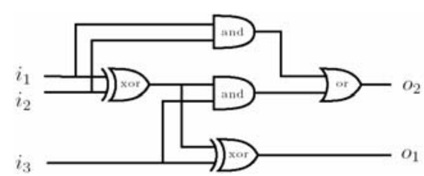
Proof System BOOL (1/2)

EQU 1 $x = y, x = 1 \rightarrow y = 1$
EQU 2 $x = y, y = 1 \rightarrow x = 1$
EQU 3 $x = y, x = 0 \rightarrow y = 0$
EQU 4 $x = y, y = 0 \rightarrow x = 0$
NOT 1 $\neg x = y, x = 1 \rightarrow y = 0$
NOT 2 $\neg x = y, x = 0 \rightarrow y = 1$
NOT 3 $\neg x = y, y = 1 \rightarrow x = 0$
NOT 4 $\neg x = y, y = 0 \rightarrow x = 1$
AND 1 $x \wedge y = z, x = 1, y = 1 \rightarrow z = 1$
AND 2 $x \wedge y = z, x = 1, z = 0 \rightarrow y = 0$
AND 3 $x \wedge y = z, y = 1, z = 0 \rightarrow x = 0$
AND 4 $x \wedge y = z, x = 0 \rightarrow z = 0$
AND 5 $x \wedge y = z, y = 0 \rightarrow z = 0$
AND 6 $x \wedge y = z, z = 1 \rightarrow x = 1, y = 1$
OR 1 $x \vee y = z, x = 1 \rightarrow z = 1$
OR 2 $x \vee y = z, x = 0, y = 0 \rightarrow z = 0$
OR 3 $x \vee y = z, x = 0, z = 1 \rightarrow y = 1$
OR 4 $x \vee y = z, y = 0, z = 1 \rightarrow x = 1$
OR 5 $x \vee y = z, y = 1 \rightarrow z = 1$
OR 6 $x \vee y = z, z = 0 \rightarrow x = 0, y = 0$

Proof System BOOL (2/2)

- **Theorem 6.3 (BOOL)** A non-failed Boolean CSP is hyper-arc consistent iff it is closed under the application of the rules of the proof system BOOL

Example: Full Adder Circuit (1/2)



$$(i_1 \oplus i_2) \oplus i_3 = o_1$$

$$(i_1 \wedge i_2) \vee (i_3 \wedge (i_1 \oplus i_2)) = o_2$$

Proof rules for XOR

XOR 1 $x \oplus y = z, x = 1, y = 1 \rightarrow z = 0$
XOR 2 $x \oplus y = z, x = 0, y = 0 \rightarrow z = 0$

Example: Full Adder Circuit (2/2)

- To show that i_1=1, i_2=1, o_1=0 follows from the assumption that i_3=0, o_2=1

$$\langle i_1 \oplus i_2 = x_1, i_1 \wedge i_2 = y_1, x_1 \oplus i_3 = o_1, \underline{i_3 \wedge x_1 = y_2}, y_1 \vee y_2 = o_2; i_3 = 0, o_2 = 1\rangle$$

AND 4

$$\langle i_1 \oplus i_2 = x_1, i_1 \wedge i_2 = y_1, x_1 \oplus i_3 = o_1, \underline{y_1 \vee y_2 = o_2}; i_3 = 0, o_2 = 1, y_2 = 0\rangle$$

OR 4

$$\langle i_1 \oplus i_2 = x_1, \underline{i_1 \wedge i_2 = y_1}, x_1 \oplus i_3 = o_1; i_3 = 0, o_2 = 1, y_2 = 0, y_1 = 1\rangle$$

AND 6

$$\langle \underline{i_1 \oplus i_2 = x_1}, x_1 \oplus i_3 = o_1; i_3 = 0, o_2 = 1, y_2 = 0, y_1 = 1, i_1 = 1, i_2 = 1\rangle$$

XOR 1

$$\langle \underline{x_1 \oplus i_3 = o_1}; i_3 = 0, o_2 = 1, y_2 = 0, y_1 = 1, i_1 = 1, i_2 = 1, x_1 = 0\rangle$$

XOR 2

$$\langle ; i_3 = 0, o_2 = 1, y_2 = 0, y_1 = 1, i_1 = 1, i_2 = 1, x_1 = 0; o_1 = 0\rangle$$

Linear Constraints on Integer Intervals

- **Linear expression**: a term in the alphabet that contains:
  - Two constants, 0 and 1
  - The unary function symbol '-'
  - Binary function symbols '+' and '-'
  - Abbreviate: $\underbrace{1+\cdots+1}_{n\ \ times}$ *to* n and $\underbrace{x+\cdots+x}_{n\ \ times}$ *to* nx
- **Linear constraint:** *s op t* where $op \in \{<, \le, =, \ne, \ge, >\}$
  - For example: $3x + 4y - 5z \le 7$
- **Integer interval:** $[a..b]$

Domain Reduction Rules for Inequality Constraints: Example

- $3x + 4y - 5z \le 7$ with $x \in [l_x..h_x], y \in [l_y..h_y], z \in [l_z..h_z]$
- Rewrite as: $x \le \dfrac{7 - 4y + 5z}{3}$

  any value of *x* that satisfies it also satisfies:

  $$x \le \left\lfloor \frac{7 - 4l_y + 5h_z}{3} \right\rfloor$$
- So $[l_x..h_x]$ can be reduced to:

  $$[l_x..\min(\left\lfloor \frac{7 - 4l_y + 5h_z}{3} \right\rfloor, h_x)]$$

Domain Reduction Rules for Inequality Constraints

LINEAR INEQUALITY 1

$$\frac{\langle \Sigma_{i \in POS} a_i x_i - \Sigma_{i \in NEG} a_i x_i \le b; x_1 \in [l_1..h_1], \ldots, x_n \in [l_n..h_n]\rangle}{\langle \Sigma_{i \in POS} a_i x_i - \Sigma_{i \in NEG} a_i x_i \le b; x_1 \in [l_1'..h_1'], \ldots, x_n \in [l_n'..h_n']\rangle}$$

where for $j \in POS$ $\quad l_j' := l_j, h_j' := \min(h_j, \lfloor \alpha_j \rfloor)$

$$\alpha_j := \frac{b - \Sigma_{i \in POS - \{j\}} a_i l_i + \Sigma_{i \in NEG} a_i h_i}{a_j}$$

and for $j \in NEG$ $\quad l_j' := \max(l_j, \lceil \beta_j \rceil), h_j' := h_j$

$$\beta_j := \frac{-b + \Sigma_{i \in POS} a_i l_i - \Sigma_{i \in NEG - \{j\}} a_i h_i}{a_j}$$

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