# 9 Logical omniscience

A formula $\psi$ logically implies a formula $\varphi$ if $\varphi$ holds whenever $\psi$ holds. A set of formulas $\Psi$ logically implies $\varphi$ if $\varphi$ holds whenever all formulas in $\Psi$ hold.

**Definition 9.1.** A set of formulas $\Psi$ logically implies $\varphi$ with respect to a class of structures $\mathcal{M}$ if, for all $M \in \mathcal{M}$ and all states $s$ in $M$, $(M, s) \models \psi$ for all $\psi \in \Psi$ implies $(M, s) \models \varphi$.

A formula $\psi$ materially implies $\varphi$ if $\psi \Rightarrow \varphi$ is valid. For example, material and logical implication coincide in the class of all Kripke structures $\mathcal{M}_n$, since $\Psi$ logically implies $\varphi$ if and only if $\wedge_{\psi \in \Psi} \psi \Rightarrow \varphi$ is valid.

Two formulas are logically equivalent if each formula logically implies the other.

There are different forms of omniscience (with respect to a class of structures $\mathcal{M}$)

**Full logical omniscience** If the agent knows every formula in $\Psi$ and if $\Psi$ logically implies $\varphi$, then the agent knows $\varphi$.

**Knowledge of valid formulas** If $\varphi$ is valid, then the agent knows $\varphi$.

**Closure under logical implication** If the agent knows $\varphi$ and if $\varphi$ logically implies $\psi$, then the agent knows $\psi$.

**Closure under logical equivalence** If the agent knows $\varphi$ and if $\varphi$ and $\psi$ are logically equivalent, then the agent knows $\psi$.

The latter three forms of omniscience are special cases of full logical omniscience. There are also types of omniscience that do not necessarily follow from full logical omniscience.

**Closure under material implication** If the agent knows $\varphi$ and $\varphi \Rightarrow \psi$, then the agent knows $\psi$.

**Closure under valid implication** If the agent knows $\varphi$ and if $\varphi \Rightarrow \psi$ is valid, then the agent knows $\psi$.

**Closure under conjunction** If the agent knows both $\varphi$ and $\psi$, then the agent knows $\varphi \wedge \psi$.

## 9.1 Explicit representation of knowledge

Full logical omniscience follows from the definition of knowledge as truth in all possible worlds. Hence, the simplest way of avoiding full logical omniscience is to define knowledge explicitly. We can do this syntactically or semantically.

### 9.1.1 The syntactical approach

The truth-value of a formula in a Kripke structure $(S, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n)$ is determined by the interpretation $\pi$ and the possibility relations $\mathcal{K}_i$. We replace these with a syntactic assignment $\sigma$. Given a state $s$ and a formula $\varphi$, $\sigma(s)(\varphi)$ is the truth-value of $\varphi$ at the state $s$.

A standard syntactic assignment $\sigma$ is a syntactic assignment that satisfies the constraints

$$\sigma(s)(\varphi) = \mathit{true} \qquad \text{iff} \qquad \sigma(s)(\neg\varphi) = \mathit{false}$$

and

$$\sigma(s)(\varphi \wedge \psi) = \mathit{true} \qquad \text{iff} \qquad \sigma(s)(\varphi) = \mathit{true} \quad \text{and} \quad \sigma(s)(\psi) = \mathit{true}$$

A syntactic structure is a pair $(S, \sigma)$, where $S$ is a set of states and $\sigma$ is a standard syntactic assignment. We define

$$(M, s) \models \varphi \qquad \text{iff} \qquad \sigma(s)(\varphi) = \mathit{true}$$

No form of logical omniscience hold in the class of all syntactic structures. In fact, the only valid formulas are instances of propositional tautologies.

### 9.1.2 The semantical approach

The semantic content of a formula can be identified with its intension, the set of states where it is true. Namely, if two formulas $\varphi$ and $\psi$ have the same intension, then

$$(M, s) \models \varphi \qquad \text{iff} \qquad (M, s) \models \psi$$

That is, they are semantically indistinguishable.

A Montague-Scott structure $M$ is a tuple $(S, \pi, \mathcal{C}_1, \ldots, \mathcal{C}_n)$, where $S$ is a set of states, $\pi$ is an interpretation, and $\mathcal{C}_i$ is a function that associates with each state of $S$ a set of subsets of $S$. The semantics is defined as follows

$$\begin{array}{lll}
(M, s) \models p & \text{iff} & \pi(s)(p) = \mathit{true} \\
(M, s) \models \varphi \wedge \psi & \text{iff} & (M, s) \models \varphi \text{ and } (M, s) \models \psi \\
(M, s) \models \neg\varphi & \text{iff} & (M, s) \not\models \varphi \\
(M, s) \models K_i\varphi & \text{iff} & \{t \mid (M, t) \models \varphi\} \in \mathcal{C}_i(s)
\end{array}$$

The intension of a formula $\varphi$ in $M$ is

$$\varphi^M = \{s \mid (M, s) \models \varphi\}$$

Thus, an agent $i$ knows $\varphi$ in state $s$ if $\varphi^M \in \mathcal{C}_i(s)$.

MS structures can be seen as a generalization of Kripke structures.

**Proposition 9.2.** *Let $M = (S, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n)$ be a Kripke structure. Let $M'$ be the MS structure $(S, \pi, \mathcal{C}_1, \ldots, \mathcal{C}_n)$, where $\mathcal{C}_i(s)$ is the set of all supersets of $\mathcal{K}_i(s) = \{t \mid \langle s, t\rangle \in \mathcal{K}_i\}$. Then,*

$$(M, s) \models \varphi \qquad \textit{iff} \qquad (M', s) \models \varphi$$

*Proof.* Fix $\varphi$ and assume $(M, s) \models \varphi$ if and only if $(M', s) \models \varphi$ for all $s \in S$. We prove $(M, s) \models K_i\varphi$ if and only if $(M', s) \models K_i\varphi$, from which the claim follows.

$(M, s) \models K_i\varphi$ iff $(M, t) \models \varphi$ for all $t$ such that $\langle s, t\rangle \in \mathcal{K}_i$ iff $(M', t) \models \varphi$ for all $t$ such that $\langle s, t\rangle \in \mathcal{K}_i$. That is, iff $\mathcal{K}_i(s) \subseteq \{t \mid (M', t) \models \varphi\}$. Consequently, $(M, s) \models K_i\varphi$ iff $\{t \mid (M', t) \models \varphi\} \in \mathcal{C}_i(s)$, which holds exactly when $(M', s) \models K_i\varphi$. □

**Theorem 9.3.** *The only form of logical omniscience that holds in MS structures is closure under logical equivalence.*

*Proof.* Let $M = (\{s, t\}, \pi, \mathcal{C}_1)$ be the MS structure where $\pi(s)(p) = \mathit{true}$, $\pi(s)(q) = \mathit{false}$, $\pi(t)(p) = \mathit{false}$, and $\pi(s)(q) = \mathit{true}$. Let $\mathcal{C}_1(s) = \big\{\{s\}, \{t\}\big\}$ and let $\mathcal{C}_1(t) = \big\{\emptyset, \{s, t\}\big\}$.

Now, $p \vee \neg p$ is a valid formula, but $(p \vee \neg p)^M = \{s, t\} \notin \mathcal{C}_1(s)$. Thus, $(M, s) \not\models K_1(p \vee \neg p)$ and knowledge of valid formulas does not hold.

Note that $q$ logically implies $p \vee q$ and that $q \Rightarrow p \vee q$ is valid. The intension $q^M = \{t\} \in \mathcal{C}_1(s)$, but $(p \vee q)^M = \{s, t\} \notin \mathcal{C}_1(s)$. Hence, neither closure under logical nor valid implication holds, and full logical omniscience is not possible.

Furthermore, $(M, s) \models K_1 p$ and $(M, s) \models K_1 q$, but $(M, s) \not\models K_1(p \wedge q)$ since $(p \wedge q)^M = \emptyset \notin \mathcal{C}_1(s)$. This means that closure under conjunction does not hold.

However, $(M, t) \models K_1(p \wedge q)$ since $\emptyset \in \mathcal{C}_1(t)$. As $(p \wedge q \Rightarrow p)^M = \{s, t\} \in \mathcal{C}_1(t)$, $(M, t) \models K_1(p \wedge q \Rightarrow p)$. Now, $(M, t) \not\models K_1 p$ and closure under material implication does not hold either.

Finally, if $\varphi$ and $\psi$ are logically equivalent, then they hold in precisely the same states. Thus, $\varphi^M = \psi^M$ for any $M$. Consequently, an agent knows $\varphi$ if and only if he also knows $\psi$. □

## 9.2 Nonstandard logic

Certain forms of logical omniscience follow from full logical omniscience under standard propositional semantics. We let knowledge be defined as truth in all possible worlds, but with respect to nonstandard propositional logic.

### 9.2.1 Nonstandard structures

A nonstandard Krikpe structure $M$ is a tuple $(S, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n, {}^*)$ such that $(S, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n)$ is a Kripke structure and ${}^*$ is a function from $S$ to itself satisfying $s^{**} = s$ for every $s \in S$ (one writes $s^*$ for ${}^*(s)$.) The semantics for negation is defined as follows.

$$(M, s) \models \neg\varphi \qquad \text{iff} \qquad (M, s^*) \not\models \varphi$$

Otherwise, the semantics is defined as for Kripke structures. The class of nonstandard structures is denoted by $\mathcal{NM}_n$.

Note that we can have both $(M, s) \models \varphi$ and $(M, s) \models \neg\varphi$. A state is incomplete with respect to $\varphi$ if neither $\varphi$ nor $\neg\varphi$ is true at the state. Otherwise, the state is complete. A state is incoherent if both $\varphi$ and $\neg\varphi$ are true, and coherent otherwise. Moreover, a state is standard if $s = s^*$. The normal semantics for negation holds at standard states.

We define disjunction and material implication in the usual way:

$$\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi)$$
$$\varphi \Rightarrow \psi = \neg\varphi \vee \psi)$$

With these definitions most connectives behave as expected.

**Proposition 9.4.** *Let $M$ be a nonstandard structure. Then,*

1. $(M, s) \models \neg\neg\varphi$ *iff* $(M, s) \models \varphi$,
2. $(M, s) \models \varphi \vee \psi$ *iff* $(M, s) \models \varphi$ *or* $(M, s) \models \psi$,
3. $(M, s) \models \neg(\varphi \wedge \psi)$ *iff* $(M, s) \models \neg\varphi \vee \neg\psi$,
4. $(M, s) \models \neg(\varphi \vee \psi)$ *iff* $(M, s) \models \neg\varphi \wedge \neg\psi$,
5. $(M, s) \models \varphi \wedge (\psi_1 \vee \psi_2)$ *iff* $(M, s) \models (\varphi \wedge \psi_1) \vee (\varphi \wedge \psi_2)$, *and*
6. $\varphi \vee (\psi_1 \wedge \psi_2)$ *iff* $(M, s) \models (\varphi \vee \psi_1) \wedge (\varphi \vee \psi_2)$.

*Proof.* We prove 1. $(M, s) \models \neg\neg\varphi$ iff $(M, s^*) \not\models \neg\varphi$ iff $(M, s^{**}) \models \varphi$ iff $(M, s) \models \varphi$ since $s^{**} = s$. □

In contrast, the connective $\Rightarrow$ does not behave as expected. For example, $p$ and $p \Rightarrow q$ can be true at a state where $q$ is false.

**Theorem 9.5.** *No formula of $\mathcal{L}_n$ is valid with respect to $\mathcal{NM}_n$. In fact, there is a nonstandard structure $M$ and two states $s$ and $t$ such that*

$$(M, s) \not\models \varphi \qquad \textit{and} \qquad (M, t) \models \varphi$$

*for all formulas $\varphi \in \mathcal{L}_n$.*

*Proof.* Let $M = (\{s, t\}, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, {}^*)$ be a nonstandard structure such that $s^* = t$, $\pi(s)(p) = \mathit{false}$ and $\pi(t)(p) = \mathit{true}$ for every primitive proposition $p$, and $\mathcal{K}_i = \{\langle s, s\rangle, \langle t, t\rangle\}$ for $i = 1, \dots, n$. By induction on the structure of formulas it follows that $(M, s) \not\models \varphi$ and $(M, t) \models \varphi$ for every formula $\varphi$. $\square$

Logical and material implication do not coincide in nonstandard structures. For instance, $\neg\neg\varphi$ logically implies $\varphi$ by Proposition 9.4, while $\neg\neg\varphi \Rightarrow \varphi$ is not valid.

Just as with ordinary Kripke structures, full logical omniscience holds. In addition, closure under conjunction holds since $\varphi$ and $\psi$ logically implies $\varphi \wedge \psi$. As expected closure under material implication does not hold. Knowledge of valid formulas holds, but only trivially as there are no valid formulas.

### 9.2.2 Strong implication

Intuitively, $\varphi \Rightarrow \psi$ should mean "if $\varphi$ is true then also $\psi$ is true." The expression $\varphi \Rightarrow \psi$ is however an abbreviation for $\neg\varphi \vee \psi$ which in the nonstandard logic do not convey the intended meaning.

We therefore define strong implication using the connective $\hookrightarrow$. Define $\varphi \hookrightarrow \psi$ as

$$(M, s) \models \varphi \hookrightarrow \psi \qquad \text{iff} \qquad (M, s) \models \varphi \quad \text{implies} \quad (M, s) \models \psi$$

Let $\mathcal{L}_n^{\hookrightarrow}$ be the language $\mathcal{L}_n$ extended with the connective $\hookrightarrow$, i.e., $\varphi \hookrightarrow \psi \in \mathcal{L}_n^{\hookrightarrow}$ if $\varphi \in \mathcal{L}_n^{\hookrightarrow}$ and $\psi \in \mathcal{L}_n^{\hookrightarrow}$. Redefine $\top$ as a fixed nonstandard tautology such as $p \hookrightarrow p$.

Strong implication is very similar to standard material implication. For instance, all types of logical omniscience holds in $\mathcal{L}_n^{\hookrightarrow}$. There are differences though: in standard propositional logic $(p \wedge \neg p) \Rightarrow q$ is valid, but in nonstandard propositional logic $(p \wedge \neg p) \hookrightarrow q$ is not.

## 9.3 Impossible worlds

Logical omniscience comes from considering knowledge as truth in all possible worlds. We augment the possible worlds with impossible worlds, where normal logic does not necessarily hold.

An impossible-worlds structure $M$ is a tuple $(S, W, \sigma, \mathcal{K}_1, \ldots, \mathcal{K}_n)$ where $W \subseteq S$ is the set of possible worlds and $\sigma$ is a syntactical assignment. We require that $\sigma$ behaves standardly in possible states. That is, if $s \in W$, then

$$\begin{aligned}
\sigma(s)(\varphi \wedge \psi) = \mathit{true} \quad &\text{iff} \quad \sigma(s)(\varphi) = \mathit{true} \text{ and } \sigma(s)(\psi) = \mathit{true}, \\
\sigma(s)(\neg\varphi) = \mathit{true} \quad &\text{iff} \quad \sigma(s)(\varphi) = \mathit{false}, \text{ and} \\
\sigma(s)(K_i\varphi) = \mathit{true} \quad &\text{iff} \quad \sigma(t)(\varphi) = \mathit{true} \text{ for all } t \text{ such that } \langle s, t\rangle \in \mathcal{K}_i
\end{aligned}$$

Notice that $\sigma$ can behave arbitrarily on impossible worlds. We write $(M, s) \models \varphi$ precisely when $\sigma(s)(\varphi) = \mathit{true}$.

A set of formulas $\Psi$ logically implies a formula $\varphi$ if for all $s \in W$, $(M, s) \models \varphi$ holds whenever $(M, s) \models \psi$ holds for all $\psi \in \Psi$. Similarly, $\varphi$ is valid in $M$ if $(M, s) \models \varphi$ for all $s \in W$.

Since agents consider impossible worlds when determining their knowledge but impossible states are not considered when determining logical implication, logical omniscience need not hold.

The impossible-worlds approach is very general. Identify a nonstandard structure $M = (S, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n, {}^*)$ with an impossible-worlds structure $M' = (S, W, \sigma, \mathcal{K}_1, \ldots, \mathcal{K}_n)$ where $\sigma(s)(\varphi) = \mathit{true}$ exactly when $(M, s) \models \varphi$ and $W = \{s \in S \mid s = s^*\}$. We define a formula in $\mathcal{L}_n$ to be standard-state valid if it is true at every standard state of every nonstandard structure. Analogously, standard-state logical implication is defined as logical implication on the standard states. The standard states are the possible state in the corresponding impossible-worlds structure.

Since $\hookrightarrow$ and $\Rightarrow$ coincide in standard states, it follows that if $\varphi$ standard-state logically implies $\psi$, then $\varphi \Rightarrow \psi$ is standard-state valid.

The tautology $\varphi \vee \neg\varphi$ is standard-state valid, but $K_i(\varphi \vee \neg\varphi)$ need not hold since the agent considers incomplete states. If the agent knows that a state is complete, then it knows all tautologies. Namely, let $p_1, \ldots, p_n$ be the primitive propositions of the formula $\varphi$ and define

$$\mathit{complete}(\varphi) = (p_1 \vee \neg p_1) \wedge \cdots \wedge (p_k \vee \neg p_k)$$

Then,

**Theorem 9.6.** *Let $\varphi$ be a tautology of standard propositional logic. Then,*

$$K_i\big(\mathit{complete}(\varphi)\big) \Rightarrow K_i\varphi$$

*is standard-state valid.*

*Proof.* Let $\varphi$ be a tautology and assume that $(M, s) \models K_i\big(\mathit{complete}(\varphi)\big)$ for a nonstandard structure $M$ and a standard state $s$. Then, $(M, t) \models p_j \vee \neg p_j$ for all state $t$ such that $\langle s,t\rangle \in \mathcal{K}_i$. Then, $(M,t) \models p_j$ or $(M,t) \models \neg p_j$ for every $t$ and $j$. Hence, $(M,t) \models \varphi$ follows by Proposition 9.4 and induction. Thus, $(M,s) \models K_i\varphi$ and $K_i\big(complete(\varphi)\big) \Rightarrow K_i\varphi$ is standard-state valid. $\square$

## 9.4 Awareness

We can combine a syntactical and a semantical approach. Define knowledge as truth in all possible worlds and make a distinction between implicit and explicit knowledge. Implicit knowledge is the normal knowledgegiven by the modal operator $K_i$, and explicit knowledge is given by a new modal operator $X_i$. The idea is to introduce an awareness operator $A_i$ and define $X_i\varphi$ as $A_i\varphi \wedge K_i\varphi$. That is, an agent explicitly knows $\varphi$ if it implicitly knows $\varphi$ and if it is aware of $\varphi$. Awareness is defined syntactically.

An awareness structure is a tuple $M = (S,\pi,\mathcal{K}_1,\ldots,\mathcal{K}_n,\mathcal{A}_1,\ldots,\mathcal{A}_n)$ where $(S,\pi,\mathcal{K}_1,\ldots,\mathcal{K}_n)$ is a Kripke structure and $\mathcal{A}_i$ is a function associating a set of formulas with every state of $S$. The semantics is extended with

$$
\begin{array}{lll}
(M,s) \models A_i\varphi & \text{iff} & \varphi \in \mathcal{A}_i(s), \text{ and} \\
(M,s) \models X_i\varphi & \text{iff} & (M,s) \models A_i\varphi \text{ and } (M,s) \models K_i\varphi
\end{array}
$$

The new language is referred to as $\mathcal{L}_n^A$.

Since truth and awareness are decoupled an agent does not explicitly know all valid formulas. For example, $\neg X_i(p \vee \neg p)$ is satisfiable since the agent need not be aware of $p \vee \neg p$.

One can capture different types of awareness by restricting the functions $\mathcal{A}_i$. For instance, if we want an agent to be aware of what it is aware of, then we would demand that $\varphi \in \mathcal{A}_i(s)$ implies $A_i\varphi \in \mathcal{A}_i(s)$. Axiomatically, we would demand that $A_i\varphi \Rightarrow A_iA_i\varphi$ holds.

It turns out that every awareness structure can be represented by an impossible-worlds structure and vice versa.