4 Proof theory

Let $\mathcal{L}_n(\Phi)$ be the set of formulas constructed from the primitive propositions in Φ and from conjunctions, negations, and the modal operators $\mathcal{K}_1, \ldots, \mathcal{K}_n$. Similarly, let $\mathcal{L}_n^D(\Phi)$ be the language constructed from $\mathcal{L}_n(\Phi)$ and the operator D_G, let $\mathcal{L}_n^C(\Phi)$ be the language constructed from $\mathcal{L}_n(\Phi)$ and the operators E_G and C_G, and let $\mathcal{L}_n^{CD}(\Phi)$ be the language constructed from $\mathcal{L}_n(\Phi)$ and D_G, E_G, and C_G.

In addition, let $\mathcal{M}_n(\Phi)$ be the class of all Kripke structures of n agents over Φ. Taking Φ to be fixed, we will hereafter refrain from mentioning it, writing instead $\mathcal{M}_n$.

For a set S, define $|S|$ as the number of elements in S, and for a formula φ, define $|\varphi|$, the length, to be the number of symbols in φ.

The formula ψ is a subformula of $\varphi \in \mathcal{L}_n$, if $\psi = \varphi$ or if ψ is a subformula of φ' or φ'' when φ is equal to $\neg\varphi'$, $K_i\varphi'$, $E_G\varphi'$, $C_G\varphi'$, $D_G\varphi'$, or $\varphi' \wedge \varphi''$. Let $Sub(\varphi)$ be the set of all subformulas of φ. Notice that $|Sub(\varphi)| \leq |\varphi|$.

Finally, we say that a formula φ is valid with respect to $\mathcal{M} \subseteq \mathcal{M}_n$, and write $\mathcal{M} \models \varphi$, if φ is valid in all structures of $\mathcal{M}$. Analogously, we say that φ is satisfiable in $\mathcal{M}$ if φ is satisfiable in some structure in $\mathcal{M}$.

4.1 Axiom systems

We will now characterize the formulas in $\mathcal{L}_n$ that are valid in $\mathcal{M}_n$. An axiom system A consists of a collection of axioms and inference rules, where an axiom is a formula and an inference rule has the form "from $\varphi_1, \ldots \varphi_k$ infer ψ." A proof in A is a sequence of formulas, each of which is either an axiom or a result of an application of an inference rule. An inference rule "from $\varphi_1, \ldots \varphi_k$ infer ψ" can be applied to a sequence s of formulas if $\varphi_1, \ldots, \varphi_k$ appear in s, thereby yielding a new sequence: s followed by ψ.

A proof is said to be a proof of the formula φ if φ is the last formula in the proof. We say that φ is provable in A and write

$$A \vdash \varphi$$

when there is a proof of φ in A.

The axiom system K_n consists of two axiom schemas and two inference rule schemas:

A1 All tautologies of propositional logic,

A2 $\big(K_i\varphi \wedge K_i(\varphi \Rightarrow \psi)\big) \Rightarrow K_i\psi$, (Distribution axiom)

R1 From φ and $\varphi \Rightarrow \psi$ infer ψ, (Modus ponens)

R2 From φ infer $K_i\varphi$. (Knowledge generalization)

To simplify our syntax we assume that the connectives $\wedge$ and $\vee$ have higher precedence than $\Rightarrow$ and $\Leftrightarrow$. For instance,

$$p \vee q \Rightarrow r \qquad \text{is read} \qquad (p \vee q) \Rightarrow r$$

Example 4.1. We prove $K_1 p \Rightarrow K_1(p \vee q)$ in K_n.

1	$p \Rightarrow p \vee q$	(A1)
2	$K_1(p \Rightarrow p \vee q)$	(R2,1)
3	$K_1 p \wedge K_1(p \Rightarrow p \vee q) \Rightarrow K_1(p \vee q)$	(A2)
4	$K_1(p \Rightarrow p \vee q) \Rightarrow \Big(\big(K_1 p \wedge K_1(p \Rightarrow p \vee q) \Rightarrow K_1(p \vee q)\big) \Rightarrow \big(K_1 p \Rightarrow K_1(p \vee q)\big)\Big)$	(A1)
5	$\big(K_1 p \wedge K_1(p \Rightarrow p \vee q) \Rightarrow K_1(p \vee q)\big) \Rightarrow \big(K_1 p \Rightarrow K_1(p \vee q)\big)$	(R1,2,4)
6	$K_1 p \Rightarrow K_1(p \vee q)$	(R1,3,5)

An axiom system A is said to be sound for a language $\mathcal{L}$ with respect to a set of structures $\mathcal{M}$ if for all $\varphi \in \mathcal{L}$

$$A \vdash \varphi \qquad \text{implies} \qquad \mathcal{M} \models \varphi$$

The system A is said to be complete for $\mathcal{L}$ with respect to $\mathcal{M}$ if for all $\varphi \in \mathcal{L}$

$$\mathcal{M} \models \varphi \qquad \text{implies} \qquad A \vdash \varphi$$

Definition 4.2. Given an axiom system A, a formula φ is A-consistent iff

$$A \nvdash \neg\varphi$$

A finite set of formulas $\{\varphi_1, \dots, \varphi_k\}$ is A-consistent iff $\varphi_1 \wedge \cdots \wedge \varphi_k$ is A-consistent, and an infinite set is A-consistent iff all of its finite subsets are A-consistent.

Observe that all subsets of a finite A-consistent set are A-consistent.

Lemma 4.3. *Assume A contains the schemas A1 and R1. If F is A-consistent and finite, then so is $F' \subset F$.*

Proof. Let $F = \{\varphi_1, \dots, \varphi_k\}$ and let $F' = \{\varphi_{i_1}, \dots, \varphi_{i_l}\}$ for $i_j \in \{1, \dots, k\}$ and $j = 1, \dots, l$. Assume that F is A-consistent and F' is not. Then,

$$A \vdash \neg(\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_l})$$

and since

$$\neg(\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_l}) \Rightarrow \neg(\varphi_1 \wedge \cdots \wedge \varphi_k)$$

is a tautology,

$$A \vdash \neg(\varphi_1 \wedge \cdots \wedge \varphi_k)$$

in contradiction with the A-consistency of F. □

A set F is a maximal A-consistent set with respect to a language $\mathcal{L}$ if F is A-consistent and for all $\varphi \in \mathcal{L} - F$, the set $F \cup \{\varphi\}$ is not A-consistent.

We will now prove that K_n is a sound and complete axiomatization for $\mathcal{L}_n$ with respect to $\mathcal{M}_n$.

Lemma 4.4. *If the axiom system A includes every instance of A1 and R1 for the countable language $\mathcal{L}$, then every A-consistent set $F \subseteq \mathcal{L}$ can be extended to a maximal A-consistent set.*

Proof. Let $F \subseteq \mathcal{L}$ be A-consistent and let $\varphi_1, \varphi_2, \dots$ be an enumeration of $\mathcal{L}$. Let $F_0 = F$ and define

$$F_{i+1} = \begin{cases} F_i \cup \{\varphi_{i+1}\} & \text{if } F_i \cup \{\varphi_{i+1}\} \text{ is } A\text{-consistent,} \\ F_i & \text{otherwise,} \end{cases}$$

for $i \geq 0$. Take

$$F_\infty = \bigcup_{i=0}^{\infty} F_i$$

Since every finite subset of F_∞ is contained in a set F_i for some i, and since F_i is A-consistent for all i, the set F_∞ is also A-consistent.

Suppose $\varphi \in \mathcal{L} - F_\infty$. As $\varphi \in \mathcal{L}$, there is a k such that $\varphi = \varphi_k$. If $F_k \cup \{\varphi_k\}$ is A-consistent, then $\varphi_k \in F_{k+1} \subseteq F_\infty$ contrary to our assumption. Hence, neither $F_k \cup \{\varphi_k\}$ nor

$$F_\infty \cup \{\varphi_k\} = F_\infty \cup \{\varphi\}$$

is A-consistent, and therefore F_∞ is a maximal A-consistent set. □

Lemma 4.5. *Assume that the axiom system A contains A1 and R1. If F is a maximal A-consistent set, then for every formula $\varphi \in \mathcal{L}$ exactly one of φ and $\neg\varphi$ is in F.*

Proof. Let F be a maximal A-consistent set. If $\varphi, \neg\varphi \in F$, then $\{\varphi, \neg\varphi\}$ is A-consistent. But this is absurd, since $\neg(\varphi \wedge \neg\varphi)$ is a tautology. Thus, not both of φ and $\neg\varphi$ can be in F.

If neither of φ and $\neg\varphi$ is in F, then $F \cup \{\varphi\}$ and $F \cup \{\neg\varphi\}$ are not A-consistent. Hence, there are $\varphi_1, \dots, \varphi_k, \dots, \varphi_l \in F$ such that

$$A \vdash \neg(\varphi_1 \wedge \dots \wedge \varphi_k \wedge \varphi)$$

and

$$A \vdash \neg(\varphi_{k+1} \wedge \dots \wedge \varphi_l \wedge \neg\varphi)$$

Since

$$\neg(\varphi_1 \wedge \dots \wedge \varphi_k \wedge \varphi) \Rightarrow \big(\neg(\varphi_{k+1} \wedge \dots \wedge \varphi_l \wedge \neg\varphi) \Rightarrow \neg(\varphi_1 \wedge \dots \wedge \varphi_l)\big)$$

is a tautology,

$$A \vdash \neg(\varphi_1 \wedge \dots \wedge \varphi_l)$$

contradicting the A-consistency of F. Therefore, exactly one of φ and $\neg\varphi$ belongs to F. □

Corollary 4.6. *Assume that A contains A1 and R1, and that F is a maximal A-consistent set. Then,*

1. $\varphi \wedge \psi \in F$ *iff* $\varphi \in F$ *and* $\psi \in F$,
2. *if* $\varphi \in F$ *and* $\varphi \Rightarrow \psi \in F$, *then* $\psi \in F$,
3. *if* $A \vdash \varphi$, *then* $\varphi \in F$.

Lemma 4.7. *Let F be a maximal K_n-consistent set. Then, there is a canonical Kripke structure M and a state s such that for all $\varphi \in \mathcal{L}_n$, $(M, s) \models \varphi$ if and only if $\varphi \in F$.*

Proof. For a set of formulas F, define

$$F/K_i = \{\varphi \mid K_i\varphi \in F\}$$

Let $M = (S, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n)$, where

$$S = \{F \mid F \text{ is a maximal } \mathrm{K}_n\text{-consistent set}\}$$

$$\pi(V)(p) = \begin{cases} \mathit{true} & \text{if } p \in F, \\ \mathit{false} & \text{if } p \notin F \end{cases}$$

$$\mathcal{K}_i = \{\langle F, G\rangle \mid F/K_i \subseteq G\}$$

Let F be a maximal K_n-consistent set. We prove that $(M, F) \models \varphi$ if and only if $\varphi \in F$ by induction on the structure of φ.

If φ is a primitive proposition, then the claim holds by definition. Assume that $(M, F) \models \varphi'$ if and only if $\varphi' \in F$ for all strict subformulas of φ. If $\varphi = \neg\varphi'$, then

$$\begin{array}{rl} (M,F) \models \varphi & \text{iff} \\ (M,F) \models \neg\varphi' & \text{iff} \\ (M,F) \not\models \varphi' & \text{iff} \\ \varphi' \notin F & \text{iff} \\ \neg\varphi' \in F & \text{iff} \\ \varphi \in F & \end{array}$$

Similarly, if $\varphi = \varphi' \wedge \varphi''$, then

$$\begin{array}{rcrl} (M,F) \models \varphi & & & \text{iff} \\ (M,F) \models \varphi' \wedge \varphi'' & & & \text{iff} \\ (M,F) \models \varphi' & \text{and} & (M,F) \models \varphi'' & \text{iff} \\ \varphi' \in F & \text{and} & \varphi'' \in F & \text{iff} \\ \varphi' \wedge \varphi'' \in F & & & \text{iff} \\ \varphi \in F & & & \end{array}$$

If $\varphi = K_i\varphi'$, then $\varphi \in F$ implies $\varphi' \in F/K_i$. If $\langle F, G\rangle \in \mathcal{K}_i$, then $\varphi' \in G$ by definition. Hence, $(M, G) \models \varphi'$ for all G such that $\langle F, G\rangle \in \mathcal{K}_i$ and $(M, F) \models K_i\varphi'$.

On the other hand, if $(M, F) \models K_i\varphi'$, then $(M, G) \models \varphi'$ for all G such that $\langle F, G\rangle \in \mathcal{K}_i$. Since $\varphi' \in G$ for every $G \supseteq F/K_i$ that is a maximal K_n-consistent set, $F/K_i \cup \{\neg\varphi'\}$ is not K_n-consistent. Thus, there are $\varphi_1, \ldots, \varphi_k \in F/K_i$ such that

$$\mathrm{K}_n \vdash \neg(\varphi_1 \wedge \cdots \wedge \varphi_k \wedge \neg\varphi')$$

from which follows

$$\mathrm{K}_n \vdash \varphi_1 \Rightarrow \big(\varphi_2 \Rightarrow \cdots \Rightarrow (\varphi_k \Rightarrow \varphi') \cdots\big)$$

In addition, by R2,

$$\mathrm{K}_n \vdash K_i\Big(\varphi_1 \Rightarrow \big(\varphi_2 \Rightarrow \cdots \Rightarrow (\varphi_k \Rightarrow \varphi')\cdots\big)\Big)$$

Noticing the proof schema

1	$K_i\psi \wedge K_i(\psi \Rightarrow \psi') \Rightarrow K_i\psi'$	(A2)
2	$\big(K_i\psi \wedge K_i(\psi \Rightarrow \psi') \Rightarrow K_i\psi'\big) \Rightarrow \big(K_i(\psi \Rightarrow \psi') \Rightarrow (K_i\psi \Rightarrow K_i\psi')\big)$	(A1)
3	$K_i(\psi \Rightarrow \psi') \Rightarrow (K_i\psi \Rightarrow K_i\psi')$	(R1,1,2)

we deduce by induction

$$\mathrm{K}_n \vdash K_i\varphi_1 \Rightarrow \big(K_i\varphi_2 \Rightarrow \cdots \Rightarrow (K_i\varphi_k \Rightarrow K_i\varphi')\cdots\big)$$

Hence,

$$K_i\varphi_1 \Rightarrow \big(K_i\varphi_2 \Rightarrow \cdots \Rightarrow (K_i\varphi_k \Rightarrow K_i\varphi')\cdots\big) \in F$$

and since $\varphi_1, \ldots, \varphi_k \in F/K_i$ implies $K_i\varphi_1, \ldots, K_i\varphi_k \in F$, repeated application of Corollary 4.6 gives

$$K_i\varphi' \in F$$

Thus, $(M, F) \models \varphi$ if and only if $\varphi \in F$ for all $\varphi \in \mathcal{L}_n$. □

Theorem 4.8. K_n *is a sound and complete axiomatization of* $\mathcal{M}_n$ *for the language* $\mathcal{L}_n$.

Proof. Soundness. By induction on the length of a proof.

Completeness. Assume that the formula $\varphi \in \mathcal{L}_n$ is K_n-consistent. Then, by Lemma 4.4, there exists a maximal K_n-consistent set such that $\varphi \in F$. Consequently, by Lemma 4.7, there is a structure M and a state s such that $(M, s) \models \varphi$. Hence, every K_n-consistent formula is satisfiable in $\mathcal{M}_n$.

Moreover, suppose $\mathcal{M}_n \models \varphi$. If $\mathrm{K}_n \nvdash \varphi$, then $\mathrm{K}_n \nvdash \neg\neg\varphi$. Hence, $\neg\varphi$ is K_n-consistent and thereby satisfiable in $\mathcal{M}_n$. Since this contradicts the validity of φ, we conclude that $\mathrm{K}_n \vdash \varphi$. □

We will now examine how properties of the $\mathcal{K}_i$ relations correspond to axioms in the proof theory.

A binary relation $\mathcal{K}$ on a set S is Euclidean, if for all $s, t, u \in S$

$$\langle s, t\rangle \in \mathcal{K} \quad \text{and} \quad \langle s, u\rangle \in \mathcal{K} \quad \text{implies} \quad \langle t, u\rangle \in \mathcal{K}$$

In addition, $\mathcal{K}$ is serial if for all $s \in S$ there is a t such that $\langle s, t\rangle \in \mathcal{K}$.

Axiom	Property	Axiom name (historical)
A2. $K_i\varphi \wedge K_i(\varphi \Rightarrow \psi) \Rightarrow K_i\psi$		Distribution (**K**)
A3. $K_i\varphi \Rightarrow \varphi$	reflexive	Knowledge (**T**)
A4. $K_i\varphi \Rightarrow K_iK_i\varphi$	transitive	Positive introspection (**4**)
A5. $\neg K_i\varphi \Rightarrow K_i\neg K_i\varphi$	Euclidean	Negative introspection (**5**)
A6. $\neg K_i\bot$	serial	Consistency (**D**)
A7. $\varphi \Rightarrow K_i\neg K_i\neg\varphi$	symmetric	

Figure 1: Axioms and their corresponding properties

Proposition 4.9. *For a binary relation $\mathcal{K}$,*

1. *If $\mathcal{K}$ is reflexive and Euclidean, then $\mathcal{K}$ is symmetric and transitive.*
2. *If $\mathcal{K}$ is symmetric and transitive, then $\mathcal{K}$ is Euclidean.*
3. *The following are equivalent:*
   (a) *$\mathcal{K}$ is reflexive, symmetric, and transitive.*
   (b) *$\mathcal{K}$ is symmetric, transitive, and serial.*
   (c) *$\mathcal{K}$ is reflexive and Euclidean.*

Proof. 1. Let $\mathcal{K} \subseteq S \times S$ be reflexive and Euclidean. If $\langle s,t\rangle \in \mathcal{K}$, then $\langle s,s\rangle \in \mathcal{K}$ as $\mathcal{K}$ is reflexive. But then $\langle t,s\rangle \in \mathcal{K}$, as $\mathcal{K}$ is Euclidean. Hence, $\mathcal{K}$ is symmetric.

If $\langle s,t\rangle \in \mathcal{K}$ and $\langle t,u\rangle \in \mathcal{K}$, then $\langle t,s\rangle \in \mathcal{K}$ as $\mathcal{K}$ is symmetric. As $\langle t,s\rangle \in \mathcal{K}$ and $\langle t,u\rangle \in \mathcal{K}$ and as $\mathcal{K}$ is Euclidean, $\langle s,u\rangle \in \mathcal{K}$. Thus, $\mathcal{K}$ is transitive.

2. Let $\mathcal{K} \subseteq S \times S$ be symmetric and transitive. If $\langle s,t\rangle \in \mathcal{K}$ and $\langle s,u\rangle \in \mathcal{K}$, then $\langle t,s\rangle \in \mathcal{K}$ by symmetry and $\langle t,u\rangle \in \mathcal{K}$ by transitivity. Hence, $\mathcal{K}$ is Euclidean.

3. When $\mathcal{K}$ is reflexive $\langle s,s\rangle \in \mathcal{K}$ for every $s \in S$. But this means that $\mathcal{K}$ is serial. Hence, (a) implies (b).

Let $\mathcal{K}$ be symmetric, transitive, and serial. Seriality implies that for every $s \in S$ there exists a $t \in S$, such that $\langle s,t\rangle \in \mathcal{K}$. Symmetry implies $\langle t,s\rangle \in \mathcal{K}$ and transitivity implies $\langle s,s\rangle \in \mathcal{K}$. Thus, $\mathcal{K}$ is reflexive. Part 2 proves that $\mathcal{K}$ is Euclidean. Hence, (b) implies (c).

Part 1 proves that (c) implies (a). $\square$

The relationship between different axioms and properties of the $\mathcal{K}_i$ relations are depicted in Figure 4.1. When discussing a certain axiom system, we often use the historical names: K_n is the axioms A1 and A2, and the rules

R1 and R2, KT_n is then K_n plus the axiom **T**, and for instance $\mathrm{KD45}_n$ is the combination of A1, R1, R2, **K**, **D**, **4**, and **5**. Moreover, the K is often left out, so KT_n is T_n. The historical names are not especially important, but we will use T_n, $\mathrm{S4}_n = \mathrm{KT4}_n$, $\mathrm{S5}_n = \mathrm{KT45}_n$, and $\mathrm{KD45}_n$.

Let $\mathcal{M}_n^r$, $\mathcal{M}_n^{rt}$, $\mathcal{M}_n^{rst}$, and $\mathcal{M}_n^{elt}$ be the classes of structures that are respectively reflexive; reflexive and transitive; reflexive, symmetric, and transitive; and Euclidean, serial, and transitive.

Theorem 4.10. *For the formulas in $\mathcal{L}_n$:*

1. T_n *is a sound and complete axiomatization of* $\mathcal{M}_n^r$.
2. $\mathrm{S4}_n$ *is a sound and complete axiomatization of* $\mathcal{M}_n^{rt}$.
3. $\mathrm{S5}_n$ *is a sound and complete axiomatization of* $\mathcal{M}_n^{rst}$.
4. $\mathrm{KD45}_n$ *is a sound and complete axiomatization of* $\mathcal{M}_n^{elt}$.

Proof sketch. The proof proceeds as in Lemma 4.7 and Theorem 4.8 with the difference that in each case one has to prove that the $\mathcal{K}_i$-relations of the canonical structure satisfy the relevant properties.

Soundness. The axiom A3 holds in reflexive structures by Theorem 3.4, the axiom A4 holds in transitive structures by Theorem 3.5, and A5 holds in symmetric and transitive structures by Theorem 3.6. This takes care of part 1, 2, and 3.

For part 4, take $M \in \mathcal{M}_n^{elt}$ and suppose $(M,s) \models \neg K_i\varphi$. Then, following the proof of Theorem 3.6, there is a state t such that $\langle s,t\rangle \in \mathcal{K}_i$ and $(M,t) \not\models \varphi$. Take any u for which $\langle s,u\rangle \in \mathcal{K}_i$. Since $\mathcal{K}_i$ is Euclidean, $\langle u,t\rangle \in \mathcal{K}_i$. Thus,

$$(M,u) \models \neg K_i\varphi \qquad \text{and} \qquad (M,s) \models K_i\neg K_i\varphi$$

and A5 holds. If $(M,s) \not\models \neg K_i\bot$, then $(M,s) \models K_i\bot$ and by seriality $(M,t) \models \bot$ for some state t. Since this is not possible, A6 holds.

Completeness. For a set of formulas F and an axiom system A, define

$$F/K_i = \{\varphi \mid K_i\varphi \in F\}$$

Let $M = (S, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n)$ be the canonical Kripke structure, where

$$\begin{aligned} S &= \{F \mid F \text{ is a maximal } A\text{-consistent set}\} \\ \pi(V)(p) &= \begin{cases} true & \text{if } p \in F, \\ false & \text{if } p \notin F \end{cases} \\ \mathcal{K}_i &= \{\langle F,G\rangle \mid F/K_i \subseteq G\} \end{aligned}$$

1. By part three of Corollary 4.6, all instances of the axiom A3 are in every maximal T_n-consistent set. Hence, by part two of Corollary 4.6, $F/K_i \subseteq F$ for every maximal T_n-consistent set F, and the canonical Kripke structure is reflexive.

2. Similarly, if $F/K_i \subseteq G$ and $G/K_i \subseteq H$, then by A4, $K_i\varphi \in F$ implies $K_iK_i\varphi \in F$. Thus, $F/K_i \subseteq (F/K_i)/K_i \subseteq H$ and the canonical Kripke structure is transitive.

3. Assume $F/K_i \subseteq G$. If $K_i\varphi \notin F$, then $\neg K_i\varphi \in F$ and by A5, $K_i\neg K_i\varphi \in F$, which implies $\neg K_i\varphi \in G$ and $K_i\varphi \notin G$. Therefore, $K_i\varphi \in G$ implies $K_i\varphi \in F$ and by A3, $\varphi \in F$. Hence, $G/K_i \subseteq F$ and the canonical structure is symmetric.

4. Suppose $F/K_i \subseteq G$ and $F/K_i \subseteq H$. By A5, $K_i\varphi \in G$ implies $K_i\varphi \in F$. Thus, $G/K_i \subseteq H$ since $F/K_i \subseteq H$, and the structure is Euclidean.

Assume there is no maximal $\mathrm{KD45}_n$-consistent set G such that $F/K_i \subseteq G$. Then, F/K_i is not $\mathrm{KD45}_n$-consistent and there exists a k and formulas $\varphi_1, \ldots, \varphi_k \in F/K_i$ such that $\{\varphi_1, \ldots, \varphi_{k-1}\}$ is $\mathrm{KD45}_n$-consistent, but $\{\varphi_1, \ldots, \varphi_k\}$ is not. Hence, $\mathrm{KD45}_n \vdash \neg(\varphi_1 \wedge \cdots \wedge \varphi_k)$ which by propositional reasoning implies, $\mathrm{KD45}_n \vdash \varphi_1 \wedge \cdots \wedge \varphi_{k-1} \Rightarrow \neg\varphi_k$ which in turn implies

$$\mathrm{KD45}_n \vdash K_i(\varphi_1 \wedge \cdots \wedge \varphi_{k-1} \Rightarrow \neg\varphi_k)$$

by R2. Thus,

$$K_i(\varphi_1 \wedge \cdots \wedge \varphi_{k-1} \Rightarrow \neg\varphi_k) \in F,$$
$$K_i(\varphi_1 \wedge \cdots \wedge \varphi_{k-1}) \in F,$$

and by A2

$$K_i(\varphi_1 \wedge \cdots \wedge \varphi_{k-1}) \wedge K_i(\varphi_1 \wedge \cdots \wedge \varphi_{k-1} \Rightarrow \neg\varphi_k) \Rightarrow K_i\neg\varphi_k \in F$$

from which we conclude $K_i\neg\varphi_k \in F$. Since also $K_i\varphi_k \in F$, we can use the tautology $\varphi_k \Rightarrow (\neg\varphi_k \Rightarrow \bot)$ and A2 to deduce $K_i(\neg\varphi_k \Rightarrow \bot) \in F$. Continuing in the same vein we obtain $K_i\bot \in F$ from $K_i(\neg\varphi_k \Rightarrow \bot) \in F$ and A2. This is a contradiction as $\neg K_i\bot \in F$ by A6. Hence, there is a maximal $\mathrm{KD45}_n$-consistent set G such that $F/K_i \subseteq G$. Seriality is thereby proved. $\square$

Theorem 4.10 and Proposition 4.9 immediately yield alternative characterizations of $\mathrm{S5}_n$, namely,

$$\mathrm{S5}_n = \mathrm{KT45}_n = \mathrm{KT5}_n = \mathrm{KD4(A7)}_n = \mathrm{KT4(A7)}_n$$

A structure M is a model of an axiom system A if every formula provable in A is valid in M. The soundness part of the preceding theorem shows that every structure in $\mathcal{M}_n^r$, $\mathcal{M}_n^{rt}$, $\mathcal{M}_n^{rst}$, and $\mathcal{M}_n^{elt}$ is a model of respectively T_n, $\mathrm{S4}_n$, $\mathrm{S5}_n$, and $\mathrm{KD45}_n$. The converse does not hold, for example

$$\begin{tikzcd} s_p \arrow[r] & \overset{p}{t} \arrow[loop right] \end{tikzcd}$$

is a model of $\mathrm{S5}_1$ that is not in $\mathcal{M}_1^{rst}$.