Exercise 11.8 Show that $f_{\neg x}$ does not have a fixed point.

$$f_{\neg x}(A) = f_{\neg}\big(f_x(A)\big) = f_{\neg}(A) = \overline{A}.$$

Thus, A is a fixed point of $f_{\neg x}$ if $A = \overline{A}$. This is only possible if the set of points in $\mathcal{I}$ and A are both empty.

Exercise 11.12 Suppose f_φ is downward continuous. Define $\varphi_0, \varphi_1, \ldots$ inductively by taking φ_0 to be true and φ_{i+1} to be $\varphi[x/\varphi_i]$. Prove that

$$(\mathcal{I}, r, m) \models \nu x[\varphi] \quad \text{iff} \quad (\mathcal{I}, r, m) \models \varphi_i \text{ for all } i \geq 0.$$

The claim is equivalent to

$$(\nu x[\varphi])^{\mathcal{I}} = \bigcap_{l \geq 0} \varphi_l^{\mathcal{I}}.$$

$(\nu x[\varphi])^{\mathcal{I}} = \mathit{gfp}\,(f_\varphi) = \bigcap_{l \geq 0} f_\varphi^l\big(\mathit{gfp}\,(f_\varphi)\big)$ since f_φ is downward continuous and therefore monotone. Since $f_\varphi(\varphi_l^{\mathcal{I}}) = (\varphi[x/\varphi_l])^{\mathcal{I}} = \varphi_{l+1}^{\mathcal{I}}$,

$$f_\varphi^l(\varphi_m^{\mathcal{I}}) = \varphi_{l+m}^{\mathcal{I}}.$$

Now, $f_\varphi^l\big(\mathit{gfp}\,(f_\varphi)\big) \subseteq f_\varphi^l(\top^{\mathcal{I}}) = f_\varphi^l(\varphi_0^{\mathcal{I}}) = \varphi_l^{\mathcal{I}}$ implies that

$$\bigcap_{l \geq 0} f_\varphi^l\big(\mathit{gfp}\,(f_\varphi)\big) \subseteq \bigcap_{l \geq 0} \varphi_l^{\mathcal{I}}.$$

On the other hand,

$$f_\varphi\left(\bigcap_{l \geq 0} \varphi_l^{\mathcal{I}}\right) = \bigcap_{l \geq 0} f_\varphi(\varphi_l^{\mathcal{I}}) = \bigcap_{l \geq 0} \varphi_{l+1}^{\mathcal{I}} = \bigcap_{l \geq 0} \varphi_l^{\mathcal{I}},$$

since f_φ is downward continuous and since $\varphi_0^{\mathcal{I}} = \top^{\mathcal{I}}$ contains all points of $\mathcal{I}$. Thus, $\bigcap_{l \geq 0} \varphi_l^{\mathcal{I}}$ is a fixed point and $\bigcap_{l \geq 0} \varphi_l^{\mathcal{I}} \subseteq \mathit{gfp}\,(f_\varphi) = (\nu x[\varphi])^{\mathcal{I}}$.

Exercise 11.14 Prove that if $\mathcal{I}$ is an interpreted system and ψ is a monotone formula, then

1. $\models \psi\big[x/\nu x[\psi]\big] \Leftrightarrow \nu x[\psi]$, and
2. if $\mathcal{I} \models \varphi \Rightarrow \psi[x/\varphi]$, then $\mathcal{I} \models \varphi \Rightarrow \nu x[\psi]$.

By definition $f_\psi\big(\mathit{gfp}\,(f_\psi)\big) = \mathit{gfp}\,(f_\psi)$ holds. Thus,

$$f_\psi\big(f_{\nu x[\psi]}(\emptyset)\big) = f_{\nu x[\psi]}(\emptyset),$$

since $f_{\nu x[\psi]}(\emptyset) = \mathit{gfp}\,(f_\psi)$.

Now, $\big(\psi\big[x/\nu x[\psi]\big]\big)^{\mathcal{I}} = f_\psi\big((\nu x[\psi])^{\mathcal{I}}\big)$. Furthermore, $(\nu x[\psi])^{\mathcal{I}} = f_{\nu x[\psi]}(\emptyset)$. Hence,

$$\big(\psi\big[x/\nu x[\psi]\big]\big)^{\mathcal{I}} = f_\psi\big(f_{\nu x[\psi]}(\emptyset)\big) = f_{\nu x[\psi]}(\emptyset) = (\nu x[\psi])^{\mathcal{I}}.$$

But then $\psi\big[x/\nu x[\psi]\big]$ and $\nu x[\psi]$ hold at precisely the same points. Thus,

$$\models \psi\big[x/\nu x[\psi]\big] \Leftrightarrow \nu x[\psi].$$

Assume $\mathcal{I} \models \varphi \Rightarrow \psi[x/\varphi]$. Then, $\varphi^{\mathcal{I}} \subseteq (\psi[x/\varphi])^{\mathcal{I}}$. Now, $(\psi[x/\varphi])^{\mathcal{I}} = f_\psi(\varphi^{\mathcal{I}})$. Thus, $\varphi^{\mathcal{I}} \subseteq f_\psi(\varphi^{\mathcal{I}})$ together with the fact that ψ is monotone imply that $\varphi^{\mathcal{I}} \subseteq \mathit{gfp}\,(f_\psi) = (\nu x[\psi])^{\mathcal{I}}$. Hence, $\mathcal{I} \models \varphi \Rightarrow \nu x[\psi]$.