Exercise 9.1 Show that if $\mathcal{M}$ is any subclass of $\mathcal{M}_n$, then we have full logical omniscience with respect to $\mathcal{M}$.

Assume that $M = (S, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n) \in \mathcal{M}$. Assume that $M \models K_i\psi$ for all $\psi \in \Psi$. Furthermore, assume that if $(M, s) \models \psi$ for all $\psi \in \Psi$, then $(M, s) \models \varphi$.

Then, $(M, s) \models K_i\psi$ for all $\psi \in \Psi$, which implies $(M, t) \models \psi$ for all $\psi \in \Psi$ and for all t such that $(s, t) \in \mathcal{K}_i$. But then, $(M, t) \models \varphi$ for all t such that $(s, t) \in \mathcal{K}_i$. Thus, $(M, s) \models K_i\varphi$.

Consequently, $M \models K_i\varphi$ holds for all $M \in \mathcal{M}$ and we have full logical omniscience with respect to $\mathcal{M}$.

Exercise 9.14(a) Show that the following are equivalent for nonstandard structures:

1. $(M, s) \models \varphi_1 \Rightarrow \varphi_2$.
2. If $(M, s^*) \models \varphi_1$, then $(M, s) \models \varphi_2$.

$$
\begin{array}{llll}
(M, s) \models \varphi_1 \Rightarrow \varphi_2 & & & \text{iff} \\
(M, s) \models \neg\varphi_1 \vee \varphi_2 & & & \text{iff} \\
(M, s) \models \neg\varphi_1 & \text{or} & (M, s) \models \varphi_2 & \text{iff} \\
(M, s^*) \not\models \varphi_1 & \text{or} & (M, s) \models \varphi_2 & \text{iff} \\
(M, s^*) \models \varphi_1 & \text{implies} & (M, s) \models \varphi_2. &
\end{array}
$$

Exercise 9.14(b) Show that $\{p, p \Rightarrow q\}$ does not logically imply q in nonstandard structures.

Let $M = (\{s, s^*\}, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n, {}^*)$ be a nonstandard structure, where $\pi(s)(p) = \mathit{true}$, $\pi(s)(q) = \mathit{false}$, and $\pi(s^*)(p) = \mathit{false}$ and $\pi(s^*)(q) = \mathit{true}$. Then, $(M, s) \models p$ and $(M, s) \models p \Rightarrow q$, but $(M, s) \not\models q$.

Note

$$
\begin{array}{lll}
(M, s) \models \varphi \vee \psi & \text{iff} & (M, s) \models \neg(\neg\varphi \wedge \neg\psi) \\
& \text{iff} & (M, s^*) \not\models \neg\varphi \wedge \neg\psi \\
& \text{iff} & (M, s^*) \not\models \neg\varphi \text{ or } (M, s^*) \not\models \neg\psi \\
& \text{iff} & (M, s^{**}) \models \varphi \text{ or } (M, s^{**}) \models \psi \\
& \text{iff} & (M, s) \models \varphi \text{ or } (M, s) \models \psi
\end{array}
$$

Exercise 9.16 Construct a nonstandard structure M and state s of M where $(M, s) \models K_i p$ and $(M, s) \models K_i(p \Rightarrow q)$, but $(M, s) \not\models K_i q$.

Let $M = (\{s, s^*\}, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n, ^*)$ be a nonstandard structure, where

$$\pi(s^*)(p) = \mathit{true}, \quad \pi(s^*)(q) = \mathit{false}, \quad \text{and} \quad \pi(s)(p) = \mathit{false}.$$

Let $\mathcal{K}_i = \{(s, s^*)\}$. Then, $(M, s) \models K_i p$ and $(M, s) \not\models K_i q$. Moreover, $(M, s^{**}) \not\models p$ implies $(M, s^*) \models \neg p$ and therefore $(M, s^*) \models p \Rightarrow q$. Thus, $(M, s) \models K_i(p \Rightarrow q)$.