Exercise 8.1(a) For all formulas φ, $\psi \in \mathcal{L}_n^{CU}$ and interpreted systems $\mathcal{I} \in \mathcal{C}_n$ prove

$$\mathcal{I} \models \bigcirc\varphi \wedge \bigcirc(\varphi \Rightarrow \psi) \Rightarrow \bigcirc\psi$$

Assume $\mathcal{I} \models \bigcirc\varphi \wedge \bigcirc(\varphi \Rightarrow \psi)$. Then, for all r and $m \geq 0$,

$$(\mathcal{I}, r, m) \models \bigcirc\varphi \wedge \bigcirc(\varphi \Rightarrow \psi)$$

implies $(\mathcal{I}, r, m+1) \models \varphi \wedge (\varphi \Rightarrow \psi)$, which implies $(\mathcal{I}, r, m+1) \models \psi)$. Thus, $(\mathcal{I}, r, m) \models \bigcirc\psi)$ for all r and m. Consequently, $\mathcal{I} \models \bigcirc\psi)$.

Exercise 8.1(b) Prove $\mathcal{I} \models \bigcirc\neg\varphi \Leftrightarrow \neg \bigcirc \varphi$.

$\mathcal{I} \models \bigcirc\neg\varphi$ iff for all r and m, $(\mathcal{I}, r, m) \models \bigcirc\neg\varphi$ iff $(\mathcal{I}, r, m+1) \models \neg\varphi$ iff $(\mathcal{I}, r, m+1) \not\models \varphi$ iff $(\mathcal{I}, r, m) \not\models \bigcirc\varphi$ iff $(\mathcal{I}, r, m) \models \neg \bigcirc \varphi$ iff $\mathcal{I} \models \neg \bigcirc \varphi$.

Exercise 8.1(c) Prove $\mathcal{I} \models \varphi U \psi \Leftrightarrow \psi \vee (\varphi \wedge \bigcirc(\varphi U \psi))$.

Assume

$$\mathcal{I} \not\models \varphi U \psi \Leftrightarrow \psi \vee (\varphi \wedge \bigcirc(\varphi U \psi)).$$

Then, there is a point (r, m) such that $(\mathcal{I}, r, m) \models \varphi U \psi$ and $(\mathcal{I}, r, m) \not\models \psi \vee (\varphi \wedge \bigcirc(\varphi U \psi))$, or $(\mathcal{I}, r, m) \not\models \varphi U \psi$ and $(\mathcal{I}, r, m) \models \psi \vee (\varphi \wedge \bigcirc(\varphi U \psi))$.

Now, $(\mathcal{I}, r, m) \models \varphi U \psi$ implies $(\mathcal{I}, r, m') \models \psi$ for some $m' \geq m$ and $(\mathcal{I}, r, m'') \models \varphi$ for all m'' with $m \leq m'' < m'$. Since $(\mathcal{I}, r, m) \not\models \psi$, $m' > m$. Hence, $(\mathcal{I}, r, m) \models \varphi$ and $(\mathcal{I}, r, m+1) \models \varphi U \psi$.

This contradiction implies $(\mathcal{I}, r, m) \not\models \varphi U \psi$ and $(\mathcal{I}, r, m) \models \psi \vee (\varphi \wedge \bigcirc(\varphi U \psi))$. Moreover, $(\mathcal{I}, r, m) \models \psi$ implies $(\mathcal{I}, r, m) \models \varphi U \psi$. Therefore, $(\mathcal{I}, r, m) \models \varphi \wedge \bigcirc(\varphi U \psi)$, which is impossible since $(\mathcal{I}, r, m) \not\models \varphi U \psi$.

Exercise 8.1(d) Prove that if $\mathcal{I} \models \varphi$ then $\mathcal{I} \models \bigcirc\varphi$.

$\mathcal{I} \models \varphi$ implies $(\mathcal{I}, r, m) \models \varphi$ for all r and m. In particular, $(\mathcal{I}, r, m+1) \models \varphi$. Thus, $(\mathcal{I}, r, m) \models \bigcirc\varphi$ for all r and m. Therefore, $\mathcal{I} \models \bigcirc\varphi$.

Exercise 8.6 Show that axiom KT1 can be deduced from $\mathrm{S5}_n^U + \mathrm{KT2}$.

1	$\top \mathcal{U} \neg\varphi \Leftrightarrow \neg\varphi \vee \big(\top \wedge \bigcirc(\top \mathcal{U} \neg\varphi)\big)$	(T3)
2	$\neg(\top \mathcal{U} \neg\varphi) \Leftrightarrow \varphi \wedge \neg \bigcirc (\top \mathcal{U} \neg\varphi)$	(Prop., 1)
3	$\neg(\top \mathcal{U} \neg\varphi) \Rightarrow \neg \bigcirc (\top \mathcal{U} \neg\varphi)$	(Prop., 2)
4	$\neg \bigcirc (\top \mathcal{U} \neg\varphi) \Leftrightarrow \bigcirc\neg(\top \mathcal{U} \neg\varphi)$	(T2)
5	$\neg(\top \mathcal{U} \neg\varphi) \Rightarrow \bigcirc\neg(\top \mathcal{U} \neg\varphi)$	(Prop., 3,4)
6	$\Box\varphi \Rightarrow \bigcirc\Box\varphi$	(Definition, 5)
7	$K_i(\Box\varphi \Rightarrow \bigcirc\Box\varphi)$	(R2, 6)
8	$K_i\Box\varphi \wedge K_i(\Box\varphi \Rightarrow \bigcirc\Box\varphi) \Rightarrow K_i \bigcirc \Box\varphi$	(A2)
9	$K_i\Box\varphi \Rightarrow K_i \bigcirc \Box\varphi$	(Prop., 7,8)

Call the last formula P1.

1	$\top \mathcal{U} \neg\varphi \Leftrightarrow \neg\varphi \vee \big(\top \wedge \bigcirc(\top \mathcal{U} \neg\varphi)\big)$	(T3)
2	$\neg(\top \mathcal{U} \neg\varphi) \Leftrightarrow \varphi \wedge \neg \bigcirc (\top \mathcal{U} \neg\varphi)$	(Prop., 1)
3	$\neg(\top \mathcal{U} \neg\varphi) \Rightarrow \varphi$	(Prop., 2)
4	$\Box\varphi \Rightarrow \varphi$	(Definition, 3)
5	$K_i(\Box\varphi \Rightarrow \varphi)$	(R2, 4)
6	$K_i\Box\varphi \wedge K_i(\Box\varphi \Rightarrow \varphi) \Rightarrow K_i\varphi$	(A2)
7	$K_i\Box\varphi \Rightarrow K_i\varphi$	(Prop., 5,6)

Call the last formula P2.

1	$K_i \bigcirc \Box\varphi \Rightarrow \bigcirc K_i\Box\varphi$	(KT2)
2	$K_i\Box\varphi \Rightarrow K_i \bigcirc \Box\varphi$	(P1)
3	$K_i\Box\varphi \Rightarrow \bigcirc K_i\Box\varphi$	(Prop., 1,2)
4	$K_i\Box\varphi \Rightarrow K_i\varphi$	(P2)
5	$K_i\Box\varphi \Rightarrow K_i\varphi \wedge \bigcirc K_i\Box\varphi$	(Prop., 3,4)
6	$K_i\Box\varphi \Rightarrow \neg(\top \mathcal{U} \neg K_i\varphi)$	(RT2, 5)
7	$K_i\Box\varphi \Rightarrow \Box K_i\varphi$	(Definition, 6)