Exercise 4.5(a) Show that for arbitrary formulas φ and ψ, we have

$$\mathcal{M}_n \models \varphi \Rightarrow \psi \qquad \text{if and only if} \qquad \mathcal{M}_n \models K_i\varphi \Rightarrow K_i\psi$$

Assume $M \models \varphi \Rightarrow \psi$ for an arbitrary $M \in \mathcal{M}_n$. Fix a state s in M. If $(M, s) \models K_i\varphi$, then for all t such that $(s, t) \in \mathcal{K}_i$, $(M, t) \models \varphi$. Since $(M, t) \models \varphi \Rightarrow \psi$ by the assumption, $(M, t) \models \psi$ for all t such that $(s, t) \in \mathcal{K}_i$. Thus, $(M, s) \models K_i\psi$. Therefore, $(M, s) \models K_i\varphi \Rightarrow K_i\psi$.

Assume $\mathcal{M}_n \models K_i\varphi \Rightarrow K_i\psi$. Suppose $\mathcal{M}_n \not\models \varphi \Rightarrow \psi$. Then, there exists an $M = (S, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n)$ in $\mathcal{M}_n$ such that $M \not\models \varphi \Rightarrow \psi$, which means that there exists a state s such that $(M, s) \not\models \varphi \Rightarrow \psi$. Thus, $(M, s) \models \varphi$ and $(M, s) \models \neg\psi$.

Now consider the Kripke structure

$$M' = (S \cup \{s'\}, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_i \cup \{(s', s)\}, \ldots, \mathcal{K}_n)$$

where $s' \notin S$. Since $(M', s) \models \varphi$, $(M', s') \models K_i\varphi$. As $(M', s') \models K_i\varphi \Rightarrow K_i\psi$, $(M', s') \models K_i\psi$, which implies $(M', s) \models \psi$. This contradicts $(M', s) \models \neg\psi$. Hence, the asumption $\mathcal{M}_n \not\models \varphi \Rightarrow \psi$ is wrong and $\mathcal{M}_n \models \varphi \Rightarrow \psi$.

Exercise 4.5(b) Show that for propositional formulas φ and ψ, we have

$$\mathcal{M}_n^{rst} \models \varphi \Rightarrow \psi \quad \text{if and only if} \quad \mathcal{M}_n^{rst} \models K_i\varphi \Rightarrow K_i\psi$$

Moreover, show that

$$\mathcal{M}_n^{rst} \models \varphi \Rightarrow \psi$$

if and only if

$$\varphi \Rightarrow \psi$$

is a propositional tautology.

If $\varphi \Rightarrow \psi$ is a propositional tautology, then $\mathcal{M}_n^{rst} \models \varphi \Rightarrow \psi$.

Assume $\mathcal{M}_n^{rst} \models \varphi \Rightarrow \psi$. Define the Kripke structure

$$M_0 = (\{s\}, \pi, \{(s, s)\}, \ldots, \{(s, s)\})$$

Clearly, $M_0 \in \mathcal{M}_n^{rst}$. Now, $M_0 \models \varphi \Rightarrow \psi$ holds for every interpretation π and thereby for every truth assignment $\pi(s)$. Hence, $\varphi \Rightarrow \psi$ is a tautology.

$\mathcal{M}_n^{rst} \models \varphi \Rightarrow \psi$ implies $\mathcal{M}_n^{rst} \models K_i\varphi \Rightarrow K_i\psi$ is showed exactly as in Exercise 4.5(a).

Assume $\mathcal{M}_n^{rst} \models K_i\varphi \Rightarrow K_i\psi$. If $\mathcal{M}_n^{rst} \not\models \varphi \Rightarrow \psi$, then $\varphi \Rightarrow \psi$ is not a tautology and there exists an interpretation π such that $(M_0, s) \not\models \varphi \Rightarrow \psi$ as

φ and ψ are propositional. Hence, $(M_0, s) \models \varphi$ and $(M_0, s) \models \neg\psi$. But then $(M_0, s) \models K_i\varphi$ holds and as $(M_0, s) \models K_i\varphi \Rightarrow K_i\psi$ holds we get $(M_0, s) \models K_i\psi$, which implies $(M_0, s) \models \psi$. This contradiction proves $\mathcal{M}_n^{rst} \models \varphi \Rightarrow \psi$

Exercise 4.5(c) Show that for arbitrary formulas φ and ψ, we have $\mathcal{M}_n^{rst} \models \varphi \Rightarrow \psi$ implies $\mathcal{M}_n^{rst} \models K_i\varphi \Rightarrow K_i\psi$, but that there exist formulas φ and ψ such that $\mathcal{M}_n^{rst} \models K_i\varphi \Rightarrow K_i\psi$, but $\mathcal{M}_n^{rst} \not\models \varphi \Rightarrow \psi$.

$\mathcal{M}_n^{rst} \models \varphi \Rightarrow \psi$ implies $\mathcal{M}_n^{rst} \models K_i\varphi \Rightarrow K_i\psi$ is showed exactly as in Exercise 4.5(a).

$\mathcal{M}_n^{rst} \models K_ip \Rightarrow K_iK_ip$ as $K_i\varphi \Rightarrow K_iK_i\varphi$ is an axiom of $S5_n$. Let $M = (S, \pi, S \times S, \ldots, S \times S)$, where $S = \{s, t\}$, $\pi(s)(p) = true$, and $\pi(t)(p) = false$. Then, $M \in \mathcal{M}_n^{rst}$ and $(M, s) \models p$, but $(M, s) \not\models K_ip$. Hence, $(M, s) \not\models p \Rightarrow K_ip$. Consequently, $\mathcal{M}_n^{rst} \not\models p \Rightarrow K_ip$.