Exercise 4.2 Prove that

$$(\mathcal{I}, r, m) \models \Diamond\varphi \qquad \text{iff} \qquad (\mathcal{I}, r, m) \models true\,\mathcal{U}\,\varphi$$

and

$$(\mathcal{I}, r, m) \models \Box\varphi \qquad \text{iff} \qquad (\mathcal{I}, r, m) \models \neg\Diamond\neg\varphi$$

$$\begin{array}{ll}
(\mathcal{I}, r, m) \models \Diamond\varphi & \text{iff} \\
(\mathcal{I}, r, m') \models \varphi \quad \text{for some } m' \geq m & \text{iff} \\
(\mathcal{I}, r, m') \models \varphi \quad \text{for some } m' \geq m \text{ and} & \\
(\mathcal{I}, r, m'') \models true \quad \text{for all } m'' \text{ with } m \leq m'' < m' & \text{iff} \\
(\mathcal{I}, r, m) \models true\,\mathcal{U}\,\varphi &
\end{array}$$

$$\begin{array}{ll}
(\mathcal{I}, r, m) \models \Box\varphi & \text{iff} \\
(\mathcal{I}, r, m') \models \varphi \quad \text{for all } m' \geq m & \text{iff} \\
(\mathcal{I}, r, m') \not\models \neg\varphi \quad \text{for all } m' \geq m & \text{iff} \\
(\mathcal{I}, r, m) \not\models \Diamond\neg\varphi & \text{iff} \\
(\mathcal{I}, r, m) \models \neg\Diamond\neg\varphi &
\end{array}$$

Exercise 4.3 Construct an interpreted system $\mathcal{I}$ and two points (r, m) and (r', m') in $\mathcal{I}$ such that $r(m) = r'(m')$, but

$$(\mathcal{I}, r, m) \models \Diamond p \qquad \text{and} \qquad (\mathcal{I}, r', m') \models \neg\Diamond p$$

Let $\mathcal{G} = \emptyset \times \{s_p, s_{\neg p}\}$, let $\mathcal{R} = \{r_1\}$ where $r_1(0) = s_{\neg p}$, $r_1(1) = s_p$, and $r_1(i) = s_{\neg p}$ for $i \geq 2$. Let

$$\pi(s_p)(p) = true$$

and let

$$\pi(s_{\neg p})(p) = false$$

Then, $\mathcal{I} = (\mathcal{R}, \pi)$ is an interpreted system.

Clearly, $r_1(0) = r_1(2)$,

$$(\mathcal{I}, r_1, 0) \models \Diamond p \qquad \text{and} \qquad (\mathcal{I}, r_1, 2) \models \neg\Diamond p$$

Exercise 4.4 Show that $(\mathcal{I}, r, m) \models \Box\Diamond\varphi$ exactly if the set

$$\{m' \mid (\mathcal{I}, r, m') \models \varphi\}$$

is infinite.

$$\begin{array}{rll} (\mathcal{I}, r, m) \models \Box\Diamond\varphi & & \text{iff} \\ (\mathcal{I}, r, m') \models \Diamond\varphi \quad \text{for all } m' \geq m & & \text{iff} \\ (\mathcal{I}, r, m'') \models \varphi \quad \text{for all } m' \geq m \text{ and for some } m'' \geq m' & & \end{array}$$

If the set $\{m' \mid (\mathcal{I}, r, m') \models \varphi\}$ is infinite, then for all $m' \geq m$ there exists an $m'' \geq m'$ such that $(\mathcal{I}, r, m'') \models \varphi$.

On the other hand, if the set $\{m' \mid (\mathcal{I}, r, m') \models \varphi\}$ is finite, then there exists an $m' \geq m$ such that for all $m'' \geq m'$, $(\mathcal{I}, r, m'') \not\models \varphi$.