**Exercise 3.1** Show that $|Sub(\varphi)| \leq |\varphi|$.

Let $\varphi$ be a formula of length one. Then,

$$|Sub(\varphi)| = 1 = |\varphi|$$

Induction hypothesis: assume that if $|\varphi| < n$, then $|Sub(\varphi)| \leq |\varphi|$.

Let $|\varphi| = n$. Then, $\varphi$ is of the form

$$\neg\varphi',\ K_i\varphi',\ C_G\varphi',\ D_G\varphi',\ E_G\varphi', \text{ or } \varphi' \wedge \varphi''$$

The number of subformulas of the five first forms is

$$|Sub(\varphi)| = 1 + |Sub(\varphi')| \leq 1 + |\varphi'| \leq |\varphi|,$$

by the induction hypothesis. The number of subformulas of the form $\varphi' \wedge \varphi''$ is

$$|Sub(\varphi)| = 1 + |Sub(\varphi')| + |Sub(\varphi'')| \leq 1 + |\varphi'| + |\varphi''| = |\varphi|.$$

Hence, induction proves that $|Sub(\varphi)| \leq |\varphi|$ for all $\varphi \in \mathcal{L}_n$.

**Exercise 3.9** Prove that A6 (consistency axiom) is provable from A3 (knowledge axiom), A1 (tautologies), and R1 (modus ponens).

| | | |
|---|---|---|
| 1 | $\top$ | (A1) |
| 2 | $K_i\bot \Rightarrow \bot$ | (A3) |
| 3 | $(K_i\bot \Rightarrow \bot) \Rightarrow (\top \Rightarrow \neg K_i\bot)$ | (A1) |
| 4 | $\top \Rightarrow \neg K_i\bot$ | (R1, 2, 3) |
| 5 | $\neg K_i\bot$ | (R1, 1, 4) |