Exercise 2.8(a) Prove $\models \big(C_G\varphi \wedge C_G(\varphi \Rightarrow \psi)\big) \Rightarrow C_G\psi$.

Suppose

$$(M, s) \models C_G\varphi \wedge C_G(\varphi \Rightarrow \psi)$$

for an arbitrary s and M. Then, $(M, t) \models \varphi$ and $(M, t) \models \varphi \Rightarrow \psi$ for all t G-reachable from s by Corollary 1.8. Hence, $(M, t) \models \psi$ by the definition of $\Rightarrow$. Thus, $(M, s) \models C_G\psi$ by Corollary 1.8 and

$$(M, s) \models \big(C_G\varphi \wedge C_G(\varphi \Rightarrow \psi)\big) \Rightarrow C_G\psi$$

holds.

But as M and s are arbitrary, the conclusion holds for all M and s. Therefore,

$$\models \big(C_G\varphi \wedge C_G(\varphi \Rightarrow \psi)\big) \Rightarrow C_G\psi$$

Exercise 2.8(b) Prove $\models C_G\varphi \Rightarrow \varphi$ assuming that the $\mathcal{K}_i$ relations are reflexive.

Assume $(M, s) \models C_G\varphi$. Then, $(M, t) \models \varphi$ for all t G-reachable from s. But then $(M, s) \models \varphi$, as s is G-reachable from s.

Exercise 2.8(c) Prove $\models C_G\varphi \Rightarrow C_GC_G\varphi$.

Let $(M, s) \models C_G\varphi$. Then, $(M, t) \models \varphi$ for all t G-reachable from s. Fix t and consider any u G-reachable from t. Since the G-reachability relation is transitive, u is G-reachable from s. Hence, $(M, u) \models \varphi$ and by Corollary 1.8 $(M, t) \models C_G\varphi$. Consequently, $(M, s) \models C_GC_G\varphi$.

Exercise 2.8(d) Prove $\models \neg C_G\varphi \Rightarrow C_G\neg C_G\varphi$ assuming that the $\mathcal{K}_i$ relations are symmetric.

Assume $(M, s) \models \neg C_G\varphi$. Then, there exists a $k \geq 1$ such that $(M, s) \models \neg E_G^k\varphi$. Thus, as the $\mathcal{K}_i$ are symmetric we have that for every state t G-reachable from s, $(M, t) \models C_G\varphi$ does not hold. This in turn implies that $(M, t) \models \neg C_G\varphi$ holds. Hence, $(M, s) \models C_G\neg C_G\varphi$ as $(M, t) \models \neg C_G\varphi$ for all t G-reachable from s.

Exercise 2.8(e) Prove $\models C_G\varphi \Rightarrow C_{G'}$ if $G' \subseteq G$.

Suppose $(M, s) \models C_G\varphi$ holds. Then, for every state t G-reachable from s, $(M, t) \models \varphi$ holds. If a state t is G'-reachable from s, then it is certainly G-reachable. Hence, for every state t G'-reachable from s, $(M, t) \models \varphi$ holds. Thus, $(M, s) \models C_{G'}\varphi$.