6.1 a) First, we construct the incidence matrix N. The rows of the matrix represent the places and the columns transitions. The order of rows is processing$_w$, ready$_w$, writing, ac, processing$_r$, ready$_r$ and reading. The order of columns is w_1,w_2,w_3,r_1,r_2 and r_3.

$$\mathbf{N} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Some sets of places whose contents stay invariant are processing$_w$, ready$_w$ and writing; processing$_r$, ready$_r$ and reading; and $3 *$ writing, ac and reading. The corresponding S-invariant vectors are $\iota_1 = (1\;1\;1\;0\;0\;0\;0)^T$, $\iota_2 = (0\;0\;0\;0\;1\;1\;1)^T$ and $\iota_3 = (0\;0\;3\;1\;0\;0\;1)^T$, respectively. The product $\mathbf{N}^T \cdot \iota_i = \mathbf{0}, i \in \{1, 2, 3\}$.

b) The three S-invariants presented in a) contain all places of the net. Therefore the net is covered by S-invariants.

6.2 Again, we first construct the incidence matrix N with places and transitions indexed in an order according to the subscripts.:

$$\mathbf{N} = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 3 & -5 & 0 & -1 \\ 0 & 3 & -2 & -1 \end{pmatrix}$$

The solution to the equation $N^T \cdot \iota = 0$ is a zero vector. Even though a zero vector is a positive S-invariant, no place is contained in it. Thus, the net is not covered by S-invariants. The equation can be solved e.g. by Gaussian elimination, using any available tool.

3 The incidence matrix is, with places and transitions indexed in an order according to the subscripts:

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The basis of the matrix is $i_1 = (1\;0\;1\;0\;1\;0)^T, i_2 = (0\;1\;0\;1\;0\;1)^T$. Both are unrealizable. A loop $M_0[t_1\rangle[t_2\rangle[t_3\rangle[t_4\rangle[t_5\rangle[t_6\rangle M_0$ contains all the reachable markings and all possible occurrences of transitions of the net. Thus, neither of the presented invariants are realizable. No matter which marking we choose, all transitions have to be fired once in order to reach the marking in question. However, the T-invariants i_1 and i_2 state that only transitions t_1, t_3, t_5 and t_2, t_4, t_6, respectively, are to be fired. Thus, the T-invariants are not realizable. The linear combination $i_1 + i_2$ is realizable, though.

4 First, we construct the incidence matrix of the net:

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{pmatrix}$$

The rows represent the places in the order $i_1, w_1, a_1, i_2, w_2, a_2, p$. The columns represent the transitions in the order $s_1, s_2, b_1, b_2, e_1, e_2$.

The task is to show that the places a_1 and a_2 can never be marked at the same time. If the marking of places a_1, a_2 and p is invariant, that is, $M(a_1) + M(a_2) + M(p) = k$ in all markings, then $M(a_1) + M(a_2) \leq k$ in all markings. Thus, we must show that $A^T \cdot \iota = \mathbf{0}$. ι is the vector representing the places belonging to the S-invariant.

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By calculating the product, we see that the result is indeed a zero vector. Now we can calculate k: $M_0^T \cdot \iota = k$. By performing the calculation we get

$$k = (1\;0\;0\;1\;0\;0\;1) \cdot (0\;0\;1\;0\;0\;1\;1)^T = 1$$

from which follows that $M(a_1) + M(a_2) \leq 1$. This means that in the places a_1 and a_2 there can be at most one token at any marking.

In general, the S-invariants can be calculated by solving the matrix equation $A^T \cdot \mathbf{y} = \mathbf{0}$. From the solution one can construct the basis for S-invariants, that is, vectors from which we can construct all possible S-invariants of the net by linear combination.

One can get the other solutions by assigning $y_2 = y_3 = 0$ or $y_1 = y_3 = 0$, when the vectors will be $(1\;1\;1\;0\;0\;0\;0)^T$ and $(0\;0\;0\;1\;1\;1\;0)^T$. These three vectors form the basis of the incidence matrix A, and all S-invariants of the net can be constructed from them with linear combination.