T–79.161 Combinatorial algorithms

Exam 06.05.2005 / Haanpää

Write on each answer sheet the name and code of the course and your name, study program, student number and signature.

1. (6 p.) Let $S = \{1, 2, \ldots, n\}$. Consider subsets of S with k elements. The rank of the first subset is 0 in all orders. Let $n = 13$ and $k = 4$.
   (a) How many subsets are there?
   (b) What is the rank of the subset $\{2, 5, 9, 11\}$ in lexicographical order?
   (c) What is the rank of the subset $\{2, 5, 9, 11\}$ in colex order?
   (d) What is the rank of the subset $\{2, 5, 9, 11\}$ in colex order, when $n = 32$?

2. (6 p.) Draw all nonisomorphic connected graphs with four vertices. Number the vertices such that the automorphism groups of the six graphs are as follows, and connect the graphs with the correct groups. The identity permutation is denoted by ε.
   (a) $\{\varepsilon, (1,4)(2,3)\}$,
   (b) $\{\varepsilon, (1)(2,3,4), (1)(2,4,3), (1)(2)(3,4), (1)(3)(2,4), (1)(4)(2,3)\}$,
   (c) $\{\varepsilon, (1)(2)(3,4)\}$,
   (d) $\{\varepsilon, (1,2,3,4), (1,3)(2,4), (1,4,3,2), (1)(3)(2,4), (2)(4)(1,3), (1,4)(2,3), (1,2)(3,4)\}$,
   (e) $\mathsf{Sym}(\{1,2,3,4\})$,
   (f) $\{\varepsilon, (1)(3)(2,4), (2)(4)(1,3), (1,3)(2,4)\}$.

3. (6 p.) Cliques and graph coloring.
   (a) Define *clique* succinctly but exactly.
   (b) Give a greedy algorithm for coloring the vertices of a given graph so that no edge has two endpoints of the same color. The algorithm should aim to use relatively few colors.
   (c) If we know that a given graph can be colored with 8 colors so that no edge has two endpoints of the same color, what can we say about the size of the largest clique in the graph? What if we additionally know that there is no such coloring with 7 colors? How could the algorithm in (b) be used in searching for the largest clique in a graph?

4. (6 p.) Local search. On the second page there are sketches of two search algorithms and two neighborhood heuristics. In the descriptions of the heuristics x is a solution, $N(x)$ is the neighborhood of the solution x, f is the objective function to be minimised, h is the neighborhood heuristic used and $\mathrm{rnd}(0\ldots1)$ is an equally distributed random number r between $0 \le r < 1$. Assume also that for all x and y it holds that $x \in N(x)$ and that $x \in N(y)$ if and only if $y \in N(x)$.
   (a) (4 p.) Examine all four ways (1a, 1b, 2a, 2b) of combining a search heuristic and a neighborhood heuristic, describe how they work and evaluate their usefulness for optimisation. Which of the combinations are known by some name?
   (b) (2 p.) Choose one of the previous four combinations and describe, how it should be amended to obtain tabu search. What is the goal of the changes?

$T = T_0$
$\alpha = 0.999$
choose x at random
repeat 10000 times:
    $y = h(x)$
    if $\text{rnd}(0\ldots1) < \exp\left(\frac{f(x)-f(y)}{T}\right)$:
        $x = y$
    $T = \alpha T$

(a) Search algorithm 1

choose x at random
repeat 10000 times:
    $y = h(x)$
    $x = y$

(b) Search algorithm 2

$h(x)$:
return a random $y \in N(x)$

(c) Neighborhood heuristic a

$h(x)$:
return the $x' \in N(x)$, for which $f(x')$ is minimal

(d) Neighborhood heuristic b

Figure 1: Heuristics