T-79.159 Cryptography and Data Security

Lecture 7: ZK and Commitments

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The problem statement

- Let L be some language (set of words), let x be an (encrypted) value
- How to prove that $x \in L$ without giving out any additional information?
  - $\star$ x is positive? x is a full square? x is prime?
- General: how to prove that "I know that $x \in L$"
- After decrypting, verifier would see x and could test that $x \in L$ but it would give more information than is often necessary

Usage examples

- Familiar scenario: authentication
- Private key: x, public key: g^x
- I want to prove you that I know the discrete logarithm of g^x
- Without revealing x itself!

You already saw this scenario (identification schemes), but these schemes were not zero-knowledge

What is knowledge?

- Hard to define - it is easier to define what is *gain of knowledge*.

- I tell you $1 + 1 = 2$. Do you gain knowledge?

  ⋆ Most of you don't.

- I tell you the factors of $2^{2^{41}} - 1$. Do you gain knowledge?

Minimizing gain of knowledge

- I prove you that I know the factors of $2^{2^{41}} - 1$, without revealing them.
- I prove that two graphs G_1 and G_2 are isomorphic without revealing the isomorphism.
  - ⋆ Graph isomorphism is a well-known hard problem
- In general: I convince you that I know something, without you getting to know anything else but that I know this something
  - ⋆ $\approx$ zero-knowledge.

Knowledge$\neq$Information

Information: You are revealed an unknown object.

- Factors of $2^{2^{41}} - 1$: no new information
- Properties of information are studied in information theory

Knowledge: You are revealed results of calculations on a publicly-known object that you cannot derive by yourself.

- Factors of $2^{2^{41}} - 1$: probably new knowledge
- Factors of a randomly generated 1024-bit integer: new knowledge, assuming that factoring is hard

Zero-knowledge: Intutition

- We talk about *ZK protocols* between verifier V and prover P

- **Big intuition**: Zero-knowledge is a property of prover P:

  ⋆ Given a common input x with prover P, whatever you can calculate, based on the interaction with P, you can calculate based on x alone.

- I.e., you can simulate P.

- *Proof system*: P still manages to convince you that $x \in L$.

Preliminaries

- For formal definition of ZK, one must define an *interactive proof system* (IP system)

- IP system consists of two interactive machines that both have private
  - ⋆ (read-only) input, (read-only) random string, read-write working space, (write-only) output

- Machines can also communicate by sending messages

Preliminaries: Interactive Protocols

- A protocol takes several steps of communications, where in every step one participant sends a message to another one
- An interactive protocol IP is a pair (P, V), where at every step one participant decides, based on the previous communication, private and common inputs, and on the random string what would be the next input
- We assume that P is computationally unbounded
- V is computationally bounded

Interactive proof system

Language L has an *interactive proof system* if there is such an interactive machine V, so that

- $\exists P$, so that $\forall x \in L$, V "accepts" the common input after the IP (P, V) with probability $\geq 2/3$
- $\forall P^*$, where (P^*, V) is an IP: For all $x \notin L$, the probability that V "accepts" is $< 1/3$
- (Probabilities are taken over the coin tosses of P, V)
- Let $\mathbf{IP}$ be the set of languages that have IP proofs

Example 1: Quadratic Residues

- Recall that $\mathbb{Z}_n^* = \{0 < x < n : \mathrm{gcd}(x, n) = 1\}$.

- Quadratic residues modulo n:

$$\mathrm{QR}(n) := \{x \in \mathbb{Z}_n^* : (\exists y) y^2 \equiv x \mod n\} ,$$

  elements that have a square root modulo n

- Quadratic nonresidues:

$$\mathrm{QNR}(n) := \{x \in \mathbb{Z}_n^* : (\nexists y) y^2 \equiv x \mod n\} .$$

Example 1: Quadratic Residues

- For prime n, establishing whether $x \in \mathrm{QR}(n)$ will be trivial
- For RSA modulus $n = pq$, establishing whether $x \in \mathrm{QR}(n)$ is equivalent to factoring n
- Quadratic Residuosity Assumption (QRA): For non-prime n and random $x \in \mathbb{Z}_n$, establishing whether $x \in \mathrm{QR}(n)$ is hard
- We will assume n is not prime
  - $\star$ QRA: $x \in^? \mathrm{QR}(n)$ is hard

Example 1: IP for QNR(n)

Parameter k and common input (x, n), where $x \in \mathrm{QNR}(n)$.

- V generates k random numbers $z_i \leftarrow_R \mathbb{Z}_n^*$ and k random bits b_i, and sends to P the tuple
$$(w_1, \ldots, w_k) \ ,$$
where $w_i \leftarrow x^{1-b_i} \cdot {z_i}^2 \mod n$.

- P sends to V a tuple // P `is omnipotent`
$$(c_1, \ldots, c_k) \ ,$$
where $c_i \leftarrow 1$ iff $w_i \in \mathrm{QR}(n)$.

- V accepts that $x \in \mathrm{QNR}(n)$ iff $b_i = c_i$, $\forall i$

Correctness of example 1

- If $x \in \mathrm{QNR}(n)$ then $w_i = x^{1-b_i} \cdot {z_i}^2 \in \mathrm{QR}(n) \iff b_i = c_i$. Since an omnipotent P can always establish whether $w_i \in \mathrm{QR}(n)$, she can also return the correct b_i. Therefore, she can make V to accept with the probability 1

- If $x \in \mathrm{QR}(n)$ then w_i will be a randomly chosen quadratic residue, independently of the value of b_i. Thus the best strategy for P would be to guess b_i randomly, which means that the probability that $b_i = c_i$, $\forall i$, is $(1/2)^k$
  - ⋆ Enlarging k will decrease this probability but will also make the protocol less efficient

Example 2: Graph Nonisomorphism

- Recall: A graph G is a set of vertices $V(G)$ together with some set $E(G) \subseteq V(G) \times V(G)$ of edges.

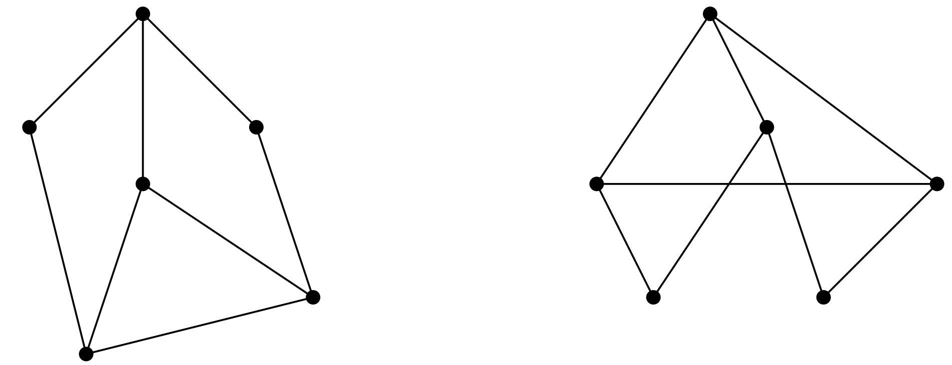
- Two graphs G_1 and G_2 are *isomorphic* if there exists an bijection $\pi : V(G_1) \to V(G_2)$, s.t.

$$(v, w) \in E(G_1) \iff (\pi(v), \pi(w)) \in E(G_2) \ .$$

  Otherwise G_1 and G_2 are nonisomorphic

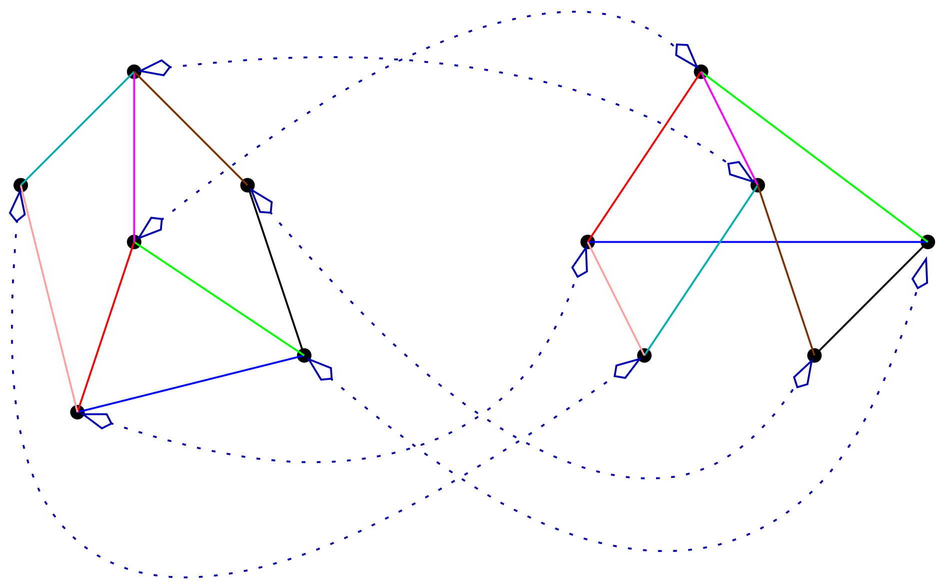
- Define GNI $:= \{(G_1, G_2) : G_1$ and G_2 are not isomorphic$\}$.

Example 2: Graph Nonisomorphism



Are these two graphs nonisomorphic?

Example 2: Graph Nonisomorphism



No! They are isomorphic: we can show an isomorphism (mapping between the nodes).

But how to show nonisomorphism? (How to convince verifier that graphs are nonisomorphic, without sending too much information?)

Example 2: Graph Nonisomorphism

- A problem is in $\mathbf{NP}$ if we know a short witness
  - $\star$ For graph isomorphism (GI), we can show π
  - $\star$ Thus $GI \in \mathbf{NP}$
- It is not known whether GNI $\in \mathbf{NP}$
- We will show that GNI $\in \mathbf{IP}$

IP for GNI

Common input (G_1, G_2). Iterate the next step for $i = 1 \ldots k$:

- V chooses a random $\alpha_i \leftarrow_R \{1, 2\}$, and a random graph G'_i from the set of graphs that are isomorphic to G_{α_i}. She sends G'_i to P
- (Omnipotent) P finds a graph G_{β_i}, s.t. G_{β_i} and G'_i are isomorphic, and sends β_i to V
  - $\star$ Intuition: P can guess α_i iff graphs are nonisomorphic

V accepts iff $\beta_i = \alpha_i$, $\forall i$

Correctness of example 2

- When $(G_1, G_2) \in$ GNI:
  - $\star$ P can distinguish isomorphic copies of graph G_1 from isomorphic copies of G_2; then V accepts with probability 1
- When $(G_1, G_2) \notin$ GNI:
  - $\star$ An isomorphic copy of G_1 is always an isomorphic copy of G_2. Thus the best strategy for P is to toss a coin, and hence the cheating probability is again $(1/2)^k$.

Back to ZK and formal definition

- Let us have an interactive proof system (P, V)
- $\text{view}_V^P(x)$ — *view* of V when interacting with P on common input x
  - ⋆ $\text{view}_V^P(x)$ is equal to the concatenation of all messages sent in this protocol, prefixed with all random coin tosses of V
- In the previous protocol:
  - ⋆ $(\alpha_1, \ldots, \alpha_k)||(G'_1, \beta_1, \ldots, G'_k, \beta_k)$

Formal definition (First try)

Definition. Let (P, V) be an IP system for language L. (P, V) is (perfect) *zero-knowledge* if for every machine (probabilistic polynomial-time) machine V^* there exists a PPT algorithm M^*, s.t. for every $x \in L$ the following two random variables are identically distributed:

- $\mathrm{view}_{V^*}^{P}(x)$ — the *view* of V^* when interacting with P.
- $M^*(x)$ — the output of M^*.

That is, $\{\mathrm{view}_{V^*}^{P}(x)\}_{x \in L} = \{M^*(x)\}_{x \in L}$ as a multiset.

Details

- Too strong a requirement! No non-trivial languages have such proofs.
- Modification: M^* can output $\perp$ with probability $\leq \frac{1}{2}$. If $M^*(x) \neq \perp$ then $\mathsf{view}^P_{V^*}(x) = M^*(x)$. (*Perfect ZK*)
- Alternate modification: $\{\mathsf{view}^P_{V^*}(x)\}_{x \in L}$ and $\{M^*(x)\}_{x \in L}$ are statistically close. (*Statistical ZK*)
- Yet another: $\{\mathsf{view}^P_{V^*}(x)\}_{x \in L}$ and $\{M^*(x)\}_{x \in L}$ cannot be distinguished in probabilistic polynomial time.

Intuition

- Perfect ZK: The distributions $\mathsf{view}_{V^*}^{P}(x)$ and $M^*(x)$ are same
- Statistical ZK: The distributions $\mathsf{view}_{V^*}^{P}(x)$ and $M^*(x)$ are close (so that even an omnipotent adversary cannot make a difference)
- Computational ZK: The distributions $\mathsf{view}_{V^*}^{P}(x)$ and $M^*(x)$ cannot be distinguished by a PPT adversary

Complexity classification

The classes of languages that have computational/statistical/perfect zero-knowledge proofs:

$$\mathbf{BPP} \subseteq_{\text{Believed that } \neq} \mathbf{PZK} \subseteq \mathbf{SZK} \subseteq_{\text{Believed that } \neq} \mathbf{CZK} = \mathbf{IP} \ .$$

$\mathbf{BPP} \subseteq \mathbf{PZK}$: Trivial, uses no interaction: $\mathbf{PZK}$ can verify by himself whether $x \in L$.

Reminder: $\mathbf{BPP}$ — set of problems that can be decided by probabilistic polynomial-time Turing machines

Example: GI $\in$ $\mathbf{PZK}$

P knows an isomorphism $\phi : G_1 \to G_2$.

1. P generates a random permutation π of G_2-s vertices. She sends $G' \leftarrow \pi(G_2)$ to V.

2. V generates a random $\sigma \leftarrow \{0, 1\}$ and sends it to P.

3. If $\sigma = 1$, P sets $\tau \leftarrow \pi \circ \phi$, otherwise she sets $\tau \leftarrow \pi$. She sends τ to V.

4. V checks that $\tau(G_\sigma) = G'$.

Intuition: $\pi(\phi(G_1)) = \phi(G_2) = G'$.

$\mathbf{NP} \subseteq \mathbf{CZK}$

- To show that there are CZK proofs for every $\mathbf{NP}$-language, it is sufficient to show a proof for one concrete $\mathbf{NP}$-complete language

- A graph G can be colored with c colors when there exists an coloring of the vertices of G with c colors so that for no edge, the vertices connected to this edge are colored with the same color

- $\chi(G)$ - the chromatic number of G. Minimum c so that G can be colored with c colors

- 3COL: the set of graphs with $\chi(G) \leq 3$. This languge is $\mathbf{NP}$-complete. Say the colors are R, G, B.

CZK protocol for $3COL$

Common input: G. P wants to prove that she knows a coloring $C : V(G \to \{R, G, B\}$ in CZK. Iterate the next protocol $|E(G)|^2$ times:

- P chooses a random permutation π of colors. She encrypts the color $\pi(C(v))$ for every vertex v, using a probabilistic public-key cryptosystem, by using a different key for every vertex. P sends to V all ciphertexts together with the correspondence between them and the vertices

- V chooses a random edge $e = (v, w)$ of the graph, and sends e to P

- P sends the decryption keys D_v and D_w to V

- V computes $\pi(C(v))$ and $\pi(C(w))$ and verifies that they are different

Correctness of this protocol

- If P knows the corresponding 3-coloring, V will never detect an incorrectly colored edge. Thus, V will accept with probability 1

- If $\chi(G) > 3$ then $\pi(C(v)) = \pi(C(w))$ in all steps with probability $\geq |E|^{-1}$. After $|E|^2$ steps the probability that V will accept is exponentially small

Reminder: Honest-Verifier ZK

- A ZK protocol is *honest-verifier*, if it is required to be ZK only in the case when the verifier follows the protocol

- Usually, in the case of HVZK protocols the verifier is only required to send random strings

- Every ZK protocol requires at least four rounds

- HVZK is achievable in 3 rounds

Non-Interactive ZK

- A ZK protocol is noninteractive, if it consists of only one step: prover sending some information to verifier

- A NIZK protocol exists only if P and V have access to some common, publicly available source of random strings (beacon)

- NIZK honest-verifier protocols exist in random-oracle model

- Many other related problems...

ZK and Commitment Schemes

- ZK: done
- Commitment schemes: next

Commitment Schemes

- P has private key K. Using this key and a random value r, she can *commit* to some x by sending $C_K(x; r)$ to V

- Later, P can reveal x and V can verify that this is the value that was previously committed

- Commitment scheme must be *hiding*: V will not be able to compute x from its commitment $C_K(x; r)$

- Commitment scheme must be *binding*: P cannot generate an $x' \neq x$, and an r', s.t. $C_K(x; r) = C_K(x'; r')$

Application: Joint coin tossing

- Alice and Bob want to decide on something by tossing a coin over a phone. How to do this securely?

- Solution: Alice commits to a random bit $b_A \leftarrow_R \{0, 1\}$, and sends $C_K(b_A; r)$ to Bob

- Bob selects a random bit $b_B \leftarrow_R \{0, 1\}$ and sends it to Alice

- Alice decommits b_A

- Alice and Bob compute the coin toss as $b_A \oplus b_B$

Pedersen commitment

Assume that $p = 2q + 1$ is a safe prime (i.e., q is also prime)

Set-up Let h be a generator of G_q, a subgroup of $\mathbb{Z}_p^*$ of prime order q. Let $g \leftarrow_R G$

- Commitment: $C_K(m; r) = g^m h^r \mod p$ where $r \leftarrow_R \mathbb{Z}_q$
- Opening: reveal m and r

Proof of security

- Unconditional hiding:
  - ⋆ Since r is a random element of $\mathbb{Z}_q$ then $g^m h^r$ is a random element of G, independently of the choice of m
- Computational binding:
  - ⋆ Given $(m; r)$, $(m'; r')$, s.t. $g^m h^r = g^{m'} h^{r'}$, $m \neq m'$, one can compute $g \leftarrow h^{(r-r')/(m'-m)}$. (This is valid since $m \neq m'$, q is prime and therefore $(m' - m)^{-1}$ exists.) Therefore, the adversary has computed the DL of g in base h
- Note that the proofs are similar to the security proofs of Schnorr's identification scheme

HVZK: protocols about commitments

Pedersen commitment scheme. Proof that P knows how to open $y = C_K(\mu;\rho)$:

- P generates a random n and a random s, and sends $a = C_K(n;s) = g^n h^s$ to V
- V generates a random $c \leftarrow \{0,1\}^t$ and sends c to P
- P sends $z = n + c\mu$, $w = s + c\rho$ to V
- Verifier checks that $C_K(z;w) \stackrel{?}{=} ay^c$.

We saw security proofs for such protocols during the last lecture

Notation

- The proof in last slide is called *proof of knowledge*
- Denoted: $PK(y = C_K(\mu; \rho))$
- Greek letters denote variables, knowledge of which is to be proved
- Other letters denote variables that are either in public knowledge or secretly owned by some party
- Another example: $PK(y = C_K(\mu; \rho) \land \mu \neq 0)$ (proof of knowledge of committed non-zero message μ)

Why commitments are good for ZK?

- Design a 3-round HVZK protocol between P and V: P sends the first and the third steps, V sends a random string on the second step.

- In practice, hard to guarantee that V does not cheat

- Solution:

  - $\star$ V selects his response c and commits to it before seeing P's first messages

  - $\star$ P sends then her first message, V opens his commitment, and P sends her second message

Advanced example: Auctions

Lipmaa, Asokan, Niemi. Secure Vickrey Auctions without Threshold Trust. Financial Cryptography 2002. Bermuda.
`http://www.tcs.hut.fi/~helger/papers/`

- You have a limited number of options: bidding $\mu \in [0, H]$

- You bid by encrypting your bid and sending it to some center

- Goal: seller S should not be able to decrypt your bid; but she should get to know the highest bid

- Solution: Encrypt by using the public key of another center A but send encryption to S

Advanced example: Auctions, 2

- Assume E is *homomorphic*: $E_K(m)E_K(m') = E_K(m + m')$
- Instead of bid μ, encrypt B^μ, where B is the maximum number of bidders
- S multiplies all ciphertexts, obtaining $c \leftarrow E_K(\sum_i B^{\mu_i})$. Due to the choice of B, this is equal to $E_K(\sum_j \alpha_j B^j)$, where α_j is the number of bidders who bid j
- S sends c to A, who decrypts c, and obtains all values α_j. A calculates the highest bid $X_1 = \max_j(\alpha_j \neq 0)$, and sends it to S
- S announces X_1 to bidders

Advanced example: Auctions, 3

- Nice protocol, but works only when different parties are honest
- Standard solution: Add a ZK proof that every step was correct
  - ⋆ Used in many cryptographic protocols!
- Every bidder proves that it encrypted a valid bid B^{μ}, $\mu \in [0, H]$
- And: A proves that A computed X_1 correctly

$PK(y = E_K(B^\mu; \rho) \wedge (\mu \in [0, H]))$

- Denote $H_j := \lfloor (H + 2^j)/2^{j+1} \rfloor$, $j = 0 \dots \lfloor \log_2 H \rfloor$. Then

$$\mu \in [0, H] \iff \mu = \sum_{j=0}^{\lfloor \log_2 H \rfloor} \mu_j H_j \quad \text{for some } \mu_j \in \{0, 1\} \ . \qquad (1)$$

- For example, $\mu \in [0, 10] \iff \mu = 5\mu_0 + 3\mu_1 + \mu_2 + \mu_3$ and $\mu \in [0, 9] \iff \mu = 5\mu_0 + 2\mu_1 + \mu_2 + \mu_3$.

- ZK proof idea: show in ZK that you know μ_j for which the right side (1) holds ("oblivious binary search")

How to prove that X_1 is correct?

- You have

$$y = E_K(\sum_j \alpha_j B^j) \ .$$

You must show that if $j > X_1$ then $\alpha_j = 0$ and if $j = X_1$ then $\alpha_j > 0$.

- Thus, this is equal to the proof that

$$PK(y = E_K(\mu; \rho) \wedge \mu = B^{X_1} + \mu_2 \wedge \mu_2 < B^{X_1+1}) \ .$$

Security properties

If A and S do not cooperate:

- A will not be able to change the highest bid or bidder
- S will not get to know anything about the bids
- A will know the statistics (how many bid j) but no individual bids
- System can be strengthened: even cooperating A and S will not be able to change the highest bid or bidder

E-voting

- E-voting: can do analogously. Bidder = voter, bid = vote
- S must get to know α_j, so instead of X_1 a ZK proof of its correctness A will send to her the sum $\sum_j \alpha_j B^j$ (simpler!)
- Problem: Can we trust that S and A do not to cooperate?
- If not, another possibility is to share the trust among a larger number of authorities

Next lecture

- Secret sharing: How to guarantee that the secret can be recovered only by priviledged sets of users?

- Threshold trust: How to guarantee in general that some system will remain secure if a majority of servers are trustworthy?

- Multi-party computation: Everything can be computed securely by using a secret-sharing approach